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INTEGRAL TRANSFORMS OF (3 M)-PARAMETRIC MULTI-INDEX MITTAG-LEFFLER FUNCTION

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ABSTRACT

The object of this paper is to evaluate the Euler, Laplace, Whittaker, K- transform and fractional Fourier transform of order α, $0 < \alpha \leq 1$, of the (3m)-parametric multi-index Mittag-Leffler function defined by Paneva-Konovska [6]. The results established in this paper would provide extensions of those given in earlier works. The result obtained is useful applied problem of science and engineering.

Mathematics Subject Classification: 33E12, 33C40

Keywords- *Parametric multi-index Mittag-Leffler function Euler transform, Laplace transform, Whittaker transform, k- transform, fractional Fourier transform, generalized wright function.*

I. INTRODUCTION

The (3m)-parametric multi-index Mittag-Leffler function introduced by Paneva-Konovska [6] is defined as

$$E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)} \frac{(z)^k}{(k!)^m}$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \ldots (\gamma_m)_k}{\Gamma(k\alpha_1+\beta_1)\ldots\Gamma(k\alpha_m+\beta_m)} \frac{(z)^k}{(k!)^m} \qquad (1)$$

$$k \in R\,;\ \alpha_i, \beta_i, \gamma_i \in C\,;\ \mathrm{Re}(\alpha_i) > 0,\ m > 1,\ i = 1, 2, \ldots m.$$

where $(\gamma)_k$ is the pochhammer symbol

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \begin{cases} 1 & ;\quad (k=0;\ \gamma \in C \setminus \{0\}) \\ \gamma(\gamma+1)\ldots(\gamma+k-1) & ;\quad (k=n \in N;\ \gamma \in C) \end{cases}$$

Particular cases

(i) For $m = 1$, equation (1) reduces to the generalized Mittag-Leffler function [7]

$$E_{\alpha,\beta}^{\gamma,1}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(k\alpha+\beta)} \frac{(z)^k}{k!} = E_{\alpha,\beta}^{\gamma}(z) \qquad (2)$$

(ii) For $\gamma = 1$, equation (2) reduces to the Mittag-Leffler function [5]

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$$E^{1}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(1)_k}{\Gamma(k\alpha+\beta)}\frac{(z)^k}{k!}=E_{\alpha,\beta}(z) \quad (3)$$

(iii) For $\beta=1,$ equation (3) reduces to the Mittag-Leffler function [4]

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{(1)_k}{\Gamma(k\alpha+1)}\frac{(z)^k}{k!}=E_{\alpha}(z) \quad (4)$$

More detail about Mittag-Leffler function and their application can be found in the paper by Saxena et al [9, 10, and 11].

The following definitions are also needed in the analysis that follows:

Definition 1

Euler Transform

The Euler transform of the function $f(z)$ is defined as

$$B\{f(z);a,b\}=\int_{0}^{1} z^{a-1}(1-z)^{b-1} f(z)\,dz \qquad a,b\in C,\operatorname{Re}(a)>0,\operatorname{Re}(b)>0 \quad (5)$$

Definition 2:

Laplace Transform

The Laplace transform of the function $f(t)$, denoted by F (s) is defined by the

equation $F(s)=(Lf)(s)=L\{f(t);s\}=\int_{0}^{\infty} e^{-st} f(t)\,dt \qquad \operatorname{Re}(s)>0 \quad (6)$

which may be symbolically written as

$$F(s)=L\{f(t);s\}\ or\ f(t)=L^{-1}\{F(s);t\}$$

provided that the function $f(t)$ is continuous for $t\geq 0$ and if exponential order as $t\to\infty$.

Definition 3 Let $u=u(t)$ be a function of the space $S(R)$, the Schwartzian space of the function that decay rapidly at infinity together with all derivatives.

The Fourier transform is given by the integral

$$\hat{u}(\omega)=\Im[u](\omega)=\int_{\mathbb{R}} U(t)\exp(i\omega t)\,dt \quad (7)$$

and the inverse Fourier transform can be defined as

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$$\Im^{-1}\left[\hat{u}\right](t) = \frac{1}{2\pi}\int_R \hat{u}(\omega)\exp(-i\omega t)\,d\omega \tag{8}$$

Definition 4 (Lizorkin space) : Let $V(R)$ be the set of functions

$$V(R)=\left\{\upsilon\in S(R) : \upsilon^{(n)}(0)=0,\ n=0,1,2,\ldots\right\}. \tag{9}$$

The Lizorkin space of function $\phi(R)$ is defined as

$$\phi(R)=\left\{\varphi\in S(R) : \Im[\varphi]\in V(R)\right\}. \tag{10}$$

Definition 5: Let u be a function belonging to $\phi(R)$.

The Fractional Fourier transform of the order α, $0<\alpha\leq 1$ is defined by

$$\hat{u}_\alpha(w) = \Im_\alpha[u](\omega) = \int_R e^{i\omega^{1/\alpha}t}u(t)\,dt \tag{11}$$

If put $\alpha=1$, equation (11) reduces to the conventional Fourier transform and for $\omega>0$, it reduces to the Fractional Fourier Transform defined by Luchko et al [2].

Lemma 1 Let u be a function of the space $\phi(R)$, let α be a real number, $0<\alpha\leq 1$, then

$$\Im_\alpha[u](\omega)=\Im[u](x),\quad for\ x=\omega^{1/\alpha} \tag{12}$$

The inverse Fractional Fourier transform of the order α, $0<\alpha\leq 1$, $u\in\phi(R)$ is defined as

$$\Im_\alpha^{-1}\left\{\hat{u}_\alpha(\omega)\right\}(t)=\frac{1}{2\pi\alpha}\int_R e^{-i\omega^{1/\alpha}t}\,\hat{u}_\alpha(\omega)\,\omega^{\frac{1-\alpha}{\alpha}}\,dw \tag{13}$$

The following result will be required in evaluating the integral (22).

$$\int_0^\infty e^{-1/2}t^{\upsilon-1}W_{\lambda,\mu}(t)\,dt=\frac{\Gamma(1/2+\mu+\upsilon)\Gamma(1/2-\mu+\upsilon)}{\Gamma(1-\lambda+\upsilon)}\qquad \mathrm{Re}(\upsilon\pm\mu)>-1/2. \tag{14}$$

where the Whittaker function $W_{\lambda,\mu}(z)$ is defined in [1](see also Mathai et al [3])

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$$W_{\mu,\upsilon}(z)=\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\lambda-\mu\right)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}M_{\lambda,-\mu}(z) \qquad (15)$$

where $M_{\lambda,\mu}(z)$ is defined as

$$M_{\lambda,\mu}(z)=z^{1/2+\mu}\;e^{-1/2z}\;{}_1F_1\left(\frac{1}{2}+\mu-\lambda\;;\;2\mu+1\;;z\right).$$

Mathai et al [3, p. 54, Eq. 2.37] defined result will be used in evaluating the integral (27).

$$\int_0^\infty t^{\rho-1}\,K_\upsilon(ax)\,dx=2^{\rho-2}\,a^{-\rho}\,\Gamma\left(\frac{\rho\pm\upsilon}{2}\right)\quad \mathrm{Re}(a)>0. \qquad (16)$$

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is defined by Wright [13-15] (also see [12, p 21]) in the following form:

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\;;Z\right]=\sum_{n=0}^{\infty}\frac{\left[\prod_{i=1}^{p}\Gamma\left(a_i+A_in\right)\right]}{\left[\prod_{j=1}^{q}\Gamma\left(b_j+B_jn\right)\right]}\frac{z^n}{n!} \qquad (17)$$

where $a_i\,,\,b_j\in C$ and $A_i\,,\,B_j\in R\;\left(i=1,\ldots,p\;;\;j=1,\ldots,q\right)$ and the defining series (17) converges for

$$\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>-1.$$

Definition 6: Let $\alpha_i>0,\;\beta_i,\gamma_i\in C\;;\;\gamma_i\neq 0,\,-1,-2,\ldots\;\; for\;i=1,2,\ldots,m$. Then the multi-index Mittag-Leffler functions (1) are Wright's generalized hypergeometric functions of the form given by [6, theorem 3, p 1093]

$$E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}(z)=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}{}_m\Psi_{2m-1}\left[\begin{matrix}(\gamma_1,1),\ldots,(\gamma_m,1)\\(\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m),(1,1),\ldots,(1,1)\end{matrix}\;;z\right] \qquad (18)$$

Theorem 1. (Euler Transform): If $k\in R\;;\;m>1;\;\alpha_i,\beta_i,\gamma_i,\eta,\delta\in C\;;\mathrm{Re}(\alpha_i)>0,\;i=1,2,\ldots,m$, then

$$\int_0^1 z^{\eta-1}\,(1-z)^{\delta-1}\,E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(xz^{\sigma}\right)dz$$

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$$= \frac{\Gamma(\delta)}{\prod_{i=1}^{m}\Gamma(\gamma_i)} \; {}_{m+1}\Psi_{2m}\left[\begin{matrix}(\gamma_1,1),\ldots(\gamma_m,1),(\eta,\sigma)\\(\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m),(1,1),\ldots,(1,1),(\eta+\delta,\sigma)\end{matrix};x\right] \quad (19)$$

Proof : - Using equation (1) and (5), it gives

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1} E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(xz^{\sigma}\right)dz$$

$$= \int_0^1 z^{\eta-1}(1-z)^{\delta-1}\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(xz^{\sigma})^k}{(k!)^m}\; dz$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(x)^k}{(k!)^m}\int_0^1 z^{\sigma k+\eta-1}(1-z)^{\delta-1}\; dz$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(x)^k}{(k!)^m}\frac{\Gamma(\sigma k+\eta)\Gamma(\delta)}{\Gamma(\sigma k+\eta+\delta)}$$

$$= \frac{\Gamma(\delta)}{\prod_{i=1}^{m}\Gamma(\gamma_i)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_i+k)\Gamma(\sigma k+\eta)}{\Gamma(k\alpha_i+\beta_i)\Gamma(\sigma k+\eta+\delta)}\frac{(x)^k}{(k!)^m}$$

This completes the proof of the Theorem 1.

Corollary 1.1 For $m=1$, equation (19) reduces in the following form given by Saxena [8]

$$\int_0^1 z^{\eta-1}(1-z)^{\delta-1}E_{\alpha,\beta}^{\gamma}\left(xz^{\sigma}\right)dz=\frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,1),(\eta,\sigma)\\(\beta,\alpha),(\eta+\delta,\sigma)\end{matrix};x\right] \quad (20)$$

Theorem 2. **(LaplaceTransform):** If $k\in R$; $m>1$; $\alpha_i,\beta_i,\gamma_i,\eta,\sigma\in C$; $\mathrm{Re}(s)>0, \mathrm{Re}(\alpha_i)>0,$

$i=1,2,\ldots,m$ *and* $\left|\frac{x}{s^{\sigma}}\right|<1,$ then

$$\int_0^{\infty} z^{\eta-1}e^{-sz}E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(x z^{\sigma}\right)dz$$

$$\frac{s^{-\eta}}{\prod_{i=1}^{m}\Gamma(\gamma_i)}\,{}_{m+1}\Psi_{2m-1}\left[\begin{matrix}(\gamma_1,1),\ldots(\gamma_m,1),(\eta,\sigma)\\(\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m)(1,1),\ldots,(1,1)\end{matrix};\frac{x}{s^{\sigma}}\right] \quad (21)$$

Proof : - Using equation (1) and (6) and gamma function formula, we obtain

$$\int_0^{\infty} z^{\eta-1}e^{-sz}E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(xz^{\sigma}\right)dz=\int_0^{\infty} z^{\eta-1}e^{-sz}\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(xz^{\sigma})^k}{(k!)^m}dz$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(x)^k}{(k!)^m}\int_0^{\infty} z^{\sigma k+\eta-1}e^{-sz}\,dz$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma_i)_k}{\Gamma(k\alpha_i + \beta_i)} \frac{(x)^k}{(k!)^m} \frac{\Gamma[\sigma k + \eta]}{s^{\sigma k + \eta}}$$

$$= \frac{s^{-\eta}}{\prod_{i=1}^{m} \Gamma(\gamma_i)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_i + k)\Gamma[\sigma k + \eta]}{\Gamma(k\alpha_i + \beta_i)\, s^{\sigma k}} \frac{(x)^k}{(k!)^m}$$

This completes the proof of the Theorem 2.

Corollary 2.1 For $m=1$ equation (21) reduces in the following form given by Saxena [8]

$$\int_0^{\infty} z^{\eta-1}\, e^{-sz}\, E_{\alpha,\beta}^{\gamma,}\left(x z^{\sigma}\right) dz = \frac{s^{-\eta}}{\Gamma(\gamma)}\, {}_2\Psi_1 \left[\begin{matrix} (\gamma, 1), (a, \sigma) \\ (\beta, \alpha) \end{matrix} ; \frac{x}{s^{\sigma}} \right] \tag{22}$$

Theorem 3. (Whittaker Transform): If $k \in R$; $m > 1$; $\alpha_i, \beta_i, \gamma_i, \rho, \delta \in C$; $\mathrm{Re}(\alpha_i) > 0$, $i = 1, 2, \ldots, m$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\rho \pm \mu) > -1/2$ *then*

$$\int_0^{\infty} t^{\rho-1}\, e^{-pt/2}\, W_{\lambda,\mu}(pt)\, E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m} \left(wt^{\delta}\right) dt$$

$$= \frac{p^{-\rho}}{\prod_{i=1}^{m} \Gamma(\gamma_i)}\, {}_{m+2}\Psi_{2m} \left[\begin{matrix} (\gamma_1, 1), \ldots (\gamma_m, 1), (1/2 \pm \mu + \rho, \delta) \\ (\beta_1, \alpha_1), \ldots, (\beta_m, \alpha_m), (1,1), \ldots, (1,1), (1 - \lambda + \rho, \delta) \end{matrix} ; \frac{w}{p^{\delta}} \right] \tag{23}$$

Proof : By virtue of equation (1) and (14), it yields

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$$\int_0^\infty t^{\rho-1}\, e^{-pt/2}\, W_{\lambda,\mu}(pt)\ E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(wt^\delta\right)dt$$

$$\int_0^\infty t^{\rho-1}\, e^{-pt/2}\, W_{\lambda,\mu}(pt)\sum_{k=0}^\infty \frac{(\gamma_i)_k\,(w)^k}{\Gamma(k\alpha_i+\beta_i)}\frac{(t)^{\delta k}}{(k!)^m}dt$$

It we set $pt=\upsilon$, then above line is equal to

$$=\int_0^\infty e^{-\upsilon/2}\left(\frac{\upsilon}{p}\right)^{\rho-1} W_{\lambda,\mu}(\upsilon)\sum_{k=0}^\infty \frac{(\gamma_i)_k\,(w)^k}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\left(\frac{\upsilon}{p}\right)^{\delta k}\frac{1}{p}\,d\upsilon$$

$$=p^{-\rho}\sum_{k=0}^\infty \frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\left(\frac{w}{p^\delta}\right)^k\int_0^\infty e^{-\upsilon/2}\,(\upsilon)^{\delta k+\rho-1}W_{\lambda,\mu}(\upsilon)d\upsilon$$

$$=p^{-\rho}\sum_{k=0}^\infty \frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\frac{\Gamma(1/2+\mu+\delta k+\rho)\,\Gamma(1/2-\mu+\delta k+\rho)}{\Gamma(1-\lambda+\delta k+\rho)}\left(\frac{w}{p^\delta}\right)^k$$

$$=\frac{p^{-\rho}}{\prod_{i=1}^m\Gamma(\gamma_i)}\sum_{k=0}^\infty \frac{\Gamma(\gamma_i+k)}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\frac{\Gamma(1/2+\mu+\delta k+\rho)\,\Gamma(1/2-\mu+\delta k+\rho)}{\Gamma(1-\lambda+\delta k+\rho)}\left(\frac{w}{p^\delta}\right)^k$$

This completes the proof of the Theorem 3.

Corollary 3.1 For $m=1$, equation (23) reduces in the following form given by Saxen [8]

$$\int_0^{\infty} t^{\rho-1}\, e^{-pt/2}\, W_{\lambda,\mu}(pt)\; E_{\alpha,\beta}^{\gamma}\left(wt^{\delta}\right)dt = \frac{p^{-\rho}}{\Gamma(\gamma)}\, {}_3\Psi_2\left[\begin{matrix}(\gamma,1),\ (1/2\pm\mu+\rho,\delta) \\ (\beta,\alpha),(1-\lambda+\rho,\delta)\end{matrix}\ ;\ \frac{w}{p^{\delta}}\right] \quad (24)$$

Fractional Fourier Transform (FFT) of multi-index (3m-Parametric)Mittag-Leffler function

Theorem 4: If $k\in R$; $m>1$; $\alpha_i,\beta_i,\gamma_i\in C$;$\mathrm{Re}(\alpha_i)>0,\ i=1,2,\ldots,m,\ 0<\alpha\le 1,$ for FFT of order ς of the multi-index (3m parametric) Mittag-Leffler function

$$\Im_{\varsigma}\left[E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\ (t)\right](\omega)=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_i+k)(i)^{-k-1}\,\omega^{-(k+1)/\varsigma}\,(-1)^{-k}\,\Gamma(k+1)}{\Gamma(k\alpha_i+\beta_i)(k!)^m} \quad (25)$$

Proof : - Using equation (7) and (11) and gamma function formula, it gives

$$\Im_{\varsigma}\left[E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\ (t)\right](\omega)=\int_R e^{i\omega^{1/\varsigma}t}\, E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\ (t)\,dt$$

$$=\int_R e^{i\omega^{1/\varsigma}t}\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)}\frac{(t)^k}{(k!)^m}\,dt$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\int_R e^{i\omega^{1/\varsigma}t}\ t^k\ dt$$

If we set $i\,\omega^{1/\varsigma}\,t=-\xi$, then

$$=\sum_{k=0}^{\infty}\frac{(\gamma_i)_k}{\Gamma(k\alpha_i+\beta_i)(k!)^m}\int_{-\infty}^{0} e^{-\xi}\left(\frac{-\xi}{i\,\omega^{1/\varsigma}}\right)^k\left(\frac{-d\xi}{i\,\omega^{1/\varsigma}}\right)$$

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$$= \sum_{k=0}^{\infty} \frac{(\gamma_i)_k}{\Gamma(k\alpha_i + \beta_i)(i)^{k+1}\omega^{(k+1)/\varsigma}(-1)^k (k!)^m} \int_0^{\infty} e^{-\xi}\xi^k d\xi$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma_i)_k \Gamma(k+1)}{\Gamma(k\alpha_i + \beta_i)(i)^{k+1}\omega^{(k+1)/\varsigma}(-1)^k (k!)^m}$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma_i)_k (i)^{-k-1}\omega^{-(k+1)/\varsigma}(-1)^{-k}\Gamma(k+1)}{\Gamma(k\alpha_i + \beta_i)(k!)^m}$$

This completes the proof of the Theorem 4.

Corollary 4.1 For $m = 1$, equation (25) reduces in the following form

$$\Im_\varsigma \left[E_{(\alpha),(\beta)}^{(\gamma)}(t) \right](\omega) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)(i)^{-k-1}\omega^{-(k+1)/\varsigma}(-1)^{-k}}{\Gamma(k\alpha+\beta)} \quad (26)$$

Theorem 5. **(K-Transform):** $k \in R$; $m > 1$; $\alpha_i, \beta_i, \gamma_i, \rho, \delta \in C$; $\operatorname{Re}(\alpha_i) > 0$, $i = 1, 2, ..., m$, then

$$\int_0^{\infty} t^{\rho-1} K_\upsilon(at) E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(xt^{2\delta}\right)dt$$

$$= \frac{2^{\rho-2}a^{-\rho}}{\prod_{i=1}^{m}\Gamma(\gamma_i)} {}_{m+2}\Psi_{2m-1}\left[\begin{matrix}(\gamma_1,1),\ldots(\gamma_m,1),\left(\frac{p\pm\upsilon}{2},\delta\right) \\ (\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m),(1,1),\ldots,(1,1),\end{matrix} ; \frac{2^{2n}x}{a^{2n}}\right] \quad (27)$$

Proof : - Using equation (1) and (16) , we obtain

$$\int_0^\infty t^{\rho-1} K_\upsilon(at) E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}\left(xt^{2\delta}\right)dt$$

$$= \int_0^\infty t^{\rho-1} K_\upsilon(at) \sum_{k=0}^\infty \frac{(\gamma_i)_k (x)^k}{\Gamma(k\alpha_i+\beta_i)} \frac{(t)^{2\delta k}}{(k!)^m} dt$$

$$= \sum_{k=0}^\infty \frac{(\gamma_i)_k (x)^k}{\Gamma(k\alpha_i+\beta_i)(k!)^m} \int_0^\infty t^{\rho+2\delta k-1} K_\upsilon(at)\, dt$$

$$= \sum_{k=0}^\infty \frac{(\gamma_i)_k (x)^k}{\Gamma(k\alpha_i+\beta_i)(k!)^m} 2^{\rho+2\delta k-2} a^{-(\rho+2\delta k)} \Gamma\left(\frac{\rho\pm\upsilon+2\delta k}{2}\right)$$

$$= 2^{\rho-2} a^{-\rho} \sum_{k=0}^\infty \frac{(\gamma_i)_k (x)^k}{\Gamma(k\alpha_i+\beta_i)(k!)^m} \Gamma\left(\frac{\rho\pm\upsilon+2\delta k}{2}\right)\left(\frac{2}{a}\right)^{2\delta k}$$

This completes the proof of the Theorem 5.

Corollary 5.3 For $m=1$, equation (27) reduces in the following form given by Saxena [8]

$$\int_0^\infty t^{\rho-1} K_\upsilon(at)\, E_{\alpha,\beta}^{\gamma}\left(xt^{2\delta}\right)dt = \frac{2^{\rho-2} a^{-\rho}}{\Gamma(\gamma)}\, {}_3\Psi_1\left[\begin{matrix}(\gamma,1),\left(\frac{\rho\pm\upsilon}{2},\delta\right) \\ (\beta,\alpha),\end{matrix} ; \frac{2^{2n} x}{a^{2n}}\right] \quad (28)$$

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