GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

THE GENERALIZED GRAVITATIONAL ENERGY MOMENTUM TENSOR

Prof. Mubarak Dirar Abd-Alla[*1] and Dr. Sawsan Ahmed Elhouri Ahmed[2]
[*1] Sudan University of Science & Technology – College of Science, Dept. of Physics. Khartoum-Sudan
[2]University of Bahri - College of Applied and Industrial Science, Dept. of Physics. Khartoum-Sudan

ABSTRACT

The generalized action based on a generalized Lagrangian, the generalized gravitational field equation is derived, a useful expression for the total energy-momentum tensor of gravity and matter is obtained by subjecting the action to variation under space-time symmetry constraint. If the Lagrangian is linear the gravitational energy disappears, but when the Lagrangian is quadratic the energy expression in the classical limit gives matter, interaction and gravitational field energy.

Keywords- *Generalized, Gravitational, Energy, Momentum, tensor.*

I. INTRODUCTION

Einstein theory of General Relativity (GR) is one of the fundamental physical theories at the present time. It describes a number of gravitational phenomena which agrees with astronomical observations [1]. Despite these successes GR suffers from being isolated from the main stream of physics. In particular, the concept of energy in GR differs radically from that in other physical theories. Namely it is devoid of the laws of conservation of matter and field combined together, and even it lacks a full expression for the energy-momentum tensor of the gravitational field [2]. Various schemes of modifications were put forward to account for this defect, but without gaining any successes. This is because these models are not in conformity with the usual notion of energy in other field theories [3,4].

On the other hand the generalized Einstein gravitational theories with additional R2 terms was used successfully to explain the problem of the flat rotation curves of galaxies [4] and inflation in the early universe without introducing fictitious scalar fields [5]. Moreover it was shown that these theories share GR all its successes in weak field. in section (2), being motivated by the successes of Generalized Field Equation (GFE) [6], a generalized energy-momentum tensor is derived from the generalized action of the GFE [7]. Unlike GR, where the energy momentum tensor is obtained by subjecting the action to the variation with respect to the field variables [8], we get the energy momentum tensor by varying the action w.r.t the coordinate variables. In this approach we relies heavily on the deep connection between the symmetries in nature and the conservation laws, where the invariance of the action under time and space transformations leads to conservation of energy momentum tensor[9].

By considering a simple non-linear Lagrangian which consists of quadratic term beside the linear one a useful expression for the energy momentum tensor is obtained in section (3). When the Lagrangian is only linear in R the gravitational part vanishes and this may explain why it is not possible to define gravity energy momentum tensor within the framework of GR. The Hamilto- nian for the quadratic Lagrangian can give in the classical limit to the matter energy, interaction and gravitational field energy.

II. SPACE-TIME SYMMETRY AND THE GRAVITATIONAL ENERGY-MOMENTUM TENSOR

The relation between the symmetry of space-time and the energy-momentum conservation is well known in physics [1]. It is generally accepted that the energy-momentum conservation law results from the invariance of the action with respect to space-time coordinates transformation [2,3,4]. Let us consider the following action integral

$$I = \int f(x^\mu, q, \partial_\lambda q, \partial_{\lambda\gamma} q) d^4x \tag{1}$$

with f defined in equation (5.1) and

$$d^4x = dx_0 dx_1 dx_2 dx_3 \tag{2}$$

and where for brevity q is set to denote the metric field $g_{\mu\nu}$ The Lagrangian density $\mathcal{L}$ depends on the scalar curvature R, which in its turn depends on both the field variable $g_{\mu\nu}$ and its first and second derivatives with respect to space-time coordinates x^μ. The variation is considered with respect to the following variables [5].

$$\begin{aligned} \delta_0 q &= q'(x) - q(x), \text{ with } , q'(x) = q(x - \delta x) \\ \delta q &= q'(x') - q(x) = q'(x') - q(x') + q(x') - q(x) \\ \delta q &= \delta_0 q + \partial_\lambda q \delta x^\lambda \end{aligned} \tag{3}$$

and

$$\begin{aligned} \delta \partial_\lambda q &= \partial_\lambda q'(x') - \partial_\lambda q(x) = \partial_\lambda \delta q \\ &= \partial_\lambda (\delta_0 q + \partial_\rho q \delta x^\rho) \\ \delta \partial_\lambda q &= \delta_0 \partial_\lambda q + \partial_{\lambda\rho} q \delta x^\rho \end{aligned} \tag{4}$$

where $\delta_0 q$ is the total variation and δq is the local variation.

The variation δI is then given by

$$\begin{aligned} \delta I &= \int_{R'} f(q + \delta_0 q, \partial_\lambda q + \delta_0 \partial_\lambda q, \partial_{\lambda\gamma} q + \delta_0 \partial_{\lambda\gamma} q) d^4x' \\ &- \int_R f(q, \partial_\lambda q, \partial_{\lambda\gamma} q) d^4x \end{aligned} \tag{5}$$

with the Jacobian

$$\frac{(\partial x')}{(\partial x)} = 1 + \frac{\partial \delta x^\lambda}{\partial x^\lambda} \tag{6}$$

and by employing partial differentiation this relation becomes

$$\begin{aligned} \delta I &= \int_R [\frac{\partial f}{\partial q} \delta_0 q + \frac{\partial f}{\partial \partial_\lambda q} \delta_0 \partial_\lambda q \\ &+ \frac{\partial f}{\partial \partial_{\lambda\gamma} q} \delta_0 \partial_{\lambda\gamma} q + f \frac{\partial \delta x^\lambda}{\partial x^\lambda}] d^4x \end{aligned} \tag{7}$$

but

$$\frac{\partial f}{\partial \partial_\lambda q} \delta_0 \partial_\lambda q = \partial_\lambda [\frac{\partial f}{\partial \partial_\lambda q} \delta_0 q] - \partial_\lambda [\frac{\partial f}{\partial \partial_\lambda q}] \delta_0 q \tag{8}$$

$$\begin{aligned}\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0\partial_{\lambda\gamma}q &= \partial_\lambda[\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0\partial_\gamma q] - \partial_\lambda[\frac{\partial f}{\partial\partial_{\lambda\gamma}q}]\delta_0\partial_\gamma q \\ &= \partial_\lambda[\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0\partial_\gamma q] - \partial_\gamma[\delta_0 q\partial_\lambda(\frac{\partial f}{\partial\partial_{\lambda\gamma}q})] \\ &+ \partial_{\lambda\gamma}\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0 q \qquad (9)\end{aligned}$$

Substituting this result in (7) yields

$$\begin{aligned}\delta I &= \int\left[\frac{\partial f}{\partial q} - \partial_\lambda\frac{\partial f}{\partial\partial_\lambda q} + \partial_{\lambda\gamma}\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\right]\delta_0 q d^4x \\ &+ \int\partial_\lambda\left[\frac{\partial f}{\partial\partial_\lambda q}\delta_0 q + \frac{\partial f}{\partial\partial_{\lambda\gamma}q}\partial_\gamma\delta_0 q\right. \\ &- \left.\delta_\gamma^\lambda\partial_\lambda\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0 q + f\delta x^\lambda\right]d^4x. \qquad (10)\end{aligned}$$

But by using the GFE the equation of motion yields [6]

$$\frac{\delta I}{\delta_0 q} = \frac{\partial f}{\partial q} - \partial_\lambda\frac{\partial f}{\partial\partial_\lambda q} + \partial_{\lambda\gamma}\frac{\partial f}{\partial\partial_{\lambda\gamma}q} = 0 \qquad (11)$$

Therefore (10) reduces to

$$\begin{aligned}\delta I &= \int\partial_\lambda[\frac{\partial f}{\partial\partial_\lambda q}\delta_0 q + \frac{\partial f}{\partial\partial_{\lambda\gamma}q}\partial_\gamma\delta_0 q \\ &- \delta_\gamma^\lambda\partial_\lambda\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\delta_0 q + f\delta x^\lambda]d^4x \qquad (12)\end{aligned}$$

Further we consider Poincare transformation of coordinates [5], i.e.

$$\delta x^\rho = \epsilon^\rho = \text{constant}. \qquad (13)$$

where by (3) and (4) we have

$$\begin{aligned}\delta_0 q &= q'(x^\rho) - q(x^\rho) = q(x^\rho - \delta x^\rho) - q(x^\rho) \\ &= q(x^\rho) - \delta x^\rho\partial_\rho q - q(x^\rho) = -\delta x^\rho\partial_\rho q \qquad (14)\end{aligned}$$

Therefore

$$\delta q = \delta_0 q + \partial_\rho q\delta x^\rho = 0 \qquad (15)$$

and

$$\delta_0\partial_\gamma q = \partial_\gamma\delta_0 q = -\partial_\gamma(\delta x^\rho\partial_\rho q) = -\delta x^\rho\partial_{\rho\gamma}q \qquad (16)$$

Hence

$$\begin{aligned}\delta I = \int\partial_\lambda([f\delta_\rho^\lambda - \frac{\partial f}{\partial\partial_\lambda q}\partial_\rho q + \delta_\gamma^\lambda\partial_\lambda\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\partial_\rho q \\ -\frac{\partial f}{\partial\partial_{\lambda\gamma}q}\partial_{\rho\gamma}q]\delta x^\rho)d^4x\end{aligned}$$

If we define the quantity within the bracket [ ] to be $\sqrt{g}V^\lambda$ therefore

$$\delta I = \int \sqrt{g}(\frac{1}{g}\partial_\lambda[\sqrt{g}V^\lambda])d^4x$$

Since δI and $\sqrt{g}d^4x$ are invariant, therefore the quantity within the bracket ( ) is a tensor. Using the identity

$$\delta I = \int \sqrt{g}V^\lambda_{;\lambda}d^4x \tag{17}$$

and recognizing that in curved space

$$d^4x \rightarrow \sqrt{g}d^4x, \ \ , \rightarrow ;$$

Therefore the energy momentum tensor T^λ_ρ is given by

$$V^\lambda = J^\lambda = -T^\lambda_\rho \delta x^\rho \tag{18}$$

and one have

$$-\sqrt{g}T^\lambda_\rho = f\delta^\lambda_\rho - \frac{\partial f}{\partial\partial_\lambda q}\partial_\rho q + \delta^\lambda_\gamma \partial_\lambda \frac{\partial f}{\partial\partial_{\lambda\gamma} q}\partial_\rho q$$
$$-\frac{\partial f}{\partial\partial_{\lambda\gamma} q}\partial_{\rho\gamma} q$$

by interchanging λ with ρ, and setting

$\gamma = \sigma$ one gets

$$\begin{aligned} -\sqrt{g}T^\rho_\lambda &= f\delta^\rho_\lambda - \frac{\partial f}{\partial\partial_\rho q}\partial_\lambda q + \partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} q}\partial_\lambda q \\ &- \frac{\partial f}{\partial\partial_{\rho\sigma} q}\partial_{\lambda\sigma} q \end{aligned} \tag{19}$$

Where from (17) V^λ is a tensor and hence T^ρ_λ is a tensor. These equations give a clear expression for the energy-momentum tensor density which normally applies to field theories other than GR. Substituting

$$q = g_{\mu\nu} \tag{20}$$

in equation (19) the following expression for the energy-momentum tensor for both metric and matter fields is obtained.

$$\begin{aligned} -\sqrt{g}T^\rho_\lambda &= f\delta^\rho_\lambda + \partial_\lambda g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}}) \\ &- \partial_{\lambda\sigma} g_{\mu\nu}\frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} \end{aligned} \tag{21}$$

To eliminate $\partial_\lambda g_{\mu\nu}$ one utilizes the relation

$$
\begin{aligned}
& \partial_\lambda g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial\partial_{\sigma\rho} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}}) \\
= \ & \delta^\rho_\lambda \partial_\rho g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}}) \\
= \ & \delta^\rho_\lambda \partial_\rho [g_{\mu\nu}\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}}] \\
- \ & \delta^\rho_\lambda g_{\mu\nu}\partial_\rho [\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}}] \\
= \ & \delta^\rho_\lambda \partial_\rho [g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}})] \\
+ \ & \delta^\rho_\lambda g_{\mu\nu} \frac{\partial f}{\partial g_{\mu\nu}} \qquad (22)
\end{aligned}
$$

The last term is simplified by using the equation of motion (11),i.e. the GFE, and it takes the following form [7]

$$
\frac{\partial f}{\partial g_{\mu\nu}} = -\partial_\rho \left(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}} \right) \qquad (23)
$$

By combining equation (22) with equation (21) the energy-momentum tensor

will be given by

$$
\begin{aligned}
-\sqrt{g} T^\rho_\lambda \ = \ & f\delta^\rho_\lambda + \delta^\rho_\lambda g_{\mu\nu} \frac{\partial f}{\partial g_{\mu\nu}} \\
+ \ & \delta^\rho_\lambda \partial_\rho \left[g_{\mu\nu} \left(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}} \right) \right] \\
& -\partial_{\lambda\sigma} g_{\mu\nu} \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} \qquad (24)
\end{aligned}
$$

Assuming that $f = \sqrt{g}\mathcal{L}$ then expressing f through the derivatives with respect to R leads to

$$
\begin{aligned}
& \delta^\rho_\lambda \partial_\rho [g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_\rho g_{\mu\nu}})] = \\
& \delta^\rho_\lambda \partial_\rho [g_{\mu\nu}(\partial_\sigma \frac{\partial f}{\partial R} \frac{\partial R}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial R} \frac{\partial R}{\partial\partial_\rho g_{\mu\nu}})] \\
& = \delta^\rho_\lambda \partial_\rho [g_{\mu\nu}\partial_\sigma (f' c^{\mu\nu\rho\sigma}) - (f' c^{\mu\nu\rho})]
\end{aligned}
$$

defining

$$
c^{\mu\nu\rho} \equiv \frac{\partial R}{\partial\partial_\rho g_{\mu\nu}}
$$

and ,

$$
c^{\mu\nu\rho\sigma} \equiv \frac{\partial R}{\partial\partial_{\rho\sigma} g_{\mu\nu}}
$$

and the expression for the scalar curvature

$$R = g^{\delta\eta} g^{\lambda\gamma}(\partial_{\gamma\lambda} g_{\delta\eta} - \partial_{\gamma\delta} g_{\lambda\eta} - \partial_{\eta\lambda} g_{\delta\gamma} + \partial_{\delta\eta} g_{\lambda\gamma})$$
$$+ g_{\kappa\alpha}(\Gamma^{\kappa}_{\gamma\lambda}\Gamma^{\alpha}_{\delta\eta} - \Gamma^{\kappa}_{\eta\lambda}\Gamma^{\alpha}_{\delta\gamma})$$

one gets

$$\begin{aligned} c^{\mu\nu\rho\sigma} &= \frac{\partial R}{\partial\partial_{\rho\sigma} g_{\mu\nu}} \\ &= g^{\lambda\gamma} g^{\delta\eta}[\delta^{\mu}_{\delta}\delta^{\nu}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\lambda} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta} \\ &- \delta^{\mu}_{\delta}\delta^{\nu}_{\gamma}\delta^{\rho}_{\eta}\delta^{\sigma}_{\lambda} + \delta^{\mu}_{\lambda}\delta^{\nu}_{\gamma}\delta^{\rho}_{\delta}\delta^{\sigma}_{\eta}] \\ &= \frac{1}{2}[g^{\rho\sigma} g^{\mu\nu} - g^{\mu\rho} g^{\nu\sigma} - g^{\nu\sigma} g^{\mu\rho} + g^{\mu\nu} g^{\rho\sigma}] \\ &= g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \end{aligned} \quad (25)$$

also []

$$c^{\mu\nu\rho} = \frac{\partial R}{\partial\partial_{\rho} g_{\mu\nu}} = 0$$

This means that the third term in the right hand side of equation (24) takes the form

$$\delta^{\rho}_{\lambda}\partial_{\rho}\left[g_{\mu\nu}\left(\partial_{\sigma}\frac{\partial f}{\partial\partial_{\rho\sigma} g_{\mu\nu}} - \frac{\partial f}{\partial\partial_{\rho} g_{\mu\nu}}\right)\right]$$
$$= \delta^{\rho}_{\lambda}\partial_{\rho}\left[g_{\mu\nu}\frac{\partial R}{\partial x^{\sigma}}\frac{\partial}{\partial R}\left(f' c^{\mu\nu\rho\sigma}\right)\right] = \delta^{\rho}_{\lambda}\partial_{\rho}\left[g_{\mu\nu} R_{;\sigma} c^{\mu\nu\rho\sigma} f''\right]$$
$$= \delta^{\rho}_{\lambda}\left[g_{\mu\nu} R_{;\sigma;\rho} c^{\mu\nu\rho\sigma} f'' + g_{\mu\nu} R_{;\sigma}\frac{\partial R}{\partial x^{\rho}}\frac{\partial}{\partial R}(f'' c^{\mu\nu\rho\sigma})\right]$$
$$= \delta^{\rho}_{\lambda}\left[g_{\mu\nu} c^{\mu\nu\rho\sigma}(R_{;\sigma;\rho} f'' + R_{;\sigma} R_{;\rho} f''')\right]$$

where the prime indicates differentiation with respect to R and by substituting $f = \sqrt{g} L$ one gets

$$\begin{aligned} & \sqrt{g}\delta^{\rho}_{\lambda}[g_{\mu\nu}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})(R_{;\sigma;\rho}\mathcal{L}'' + R_{;\sigma} R_{;\rho}\mathcal{L}''')] \\ = \;& 4\sqrt{g}\delta^{\rho}_{\lambda} g^{\rho\sigma}(R_{;\sigma;\rho}\mathcal{L}'' + R_{;\sigma} R_{;\rho}\mathcal{L}''') - \delta^{\rho}_{\nu} g^{\nu\sigma}(R_{;\sigma;\lambda}\mathcal{L}'' + R_{;\sigma} R_{;\lambda}\mathcal{L}''') \\ = \;& \sqrt{g}[\mathcal{L}''(4\delta^{\rho}_{\lambda}\Box^{2} R - g^{\rho\sigma} R_{;\lambda;\sigma}) \\ + \;& \mathcal{L}'''(4\delta^{\rho}_{\lambda} g^{\rho\sigma} R_{;\rho} R_{;\sigma} - g^{\rho\sigma} R_{;\lambda} R_{;\sigma})] \end{aligned} \quad (26)$$

Then by using the relation[]

$$R_{\mu\nu}\delta g^{\mu\nu} = -R^{\mu\nu}\delta g_{\mu\nu}$$

together with

$$\frac{\partial R}{\partial g_{\mu\nu}} = \frac{R_{\mu\nu}\partial g^{\mu\nu}}{\partial g_{\mu\nu}} = -R^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial g_{\mu\nu}} = -R^{\mu\nu}$$

yields

$$\delta^{\rho}_{\lambda} g_{\mu\nu}\frac{\partial f}{\partial g_{\mu\nu}} = g_{\mu\nu}\delta^{\rho}_{\lambda}\left(\sqrt{g}\frac{\partial \mathcal{L}}{\partial R}\frac{\partial R}{\partial g_{\mu\nu}} + \mathcal{L}\frac{\partial \sqrt{g}}{\partial g_{\mu\nu}}\right)$$

$$= \sqrt{g}\delta^{\rho}_{\lambda} g_{\mu\nu}\left[-\mathcal{L}' R^{\mu\nu} + \frac{1}{2} g^{\mu\nu}\mathcal{L}\right] = \sqrt{g}\delta^{\rho}_{\lambda}(2\mathcal{L} - \mathcal{L}' R)$$

Similarly,

$$\begin{aligned}
&\frac{\partial f}{\partial \partial_{\rho\sigma} g_{\mu\nu}}\partial_{\lambda\sigma} g_{\mu\nu} \\
&= \frac{\partial f}{\partial R}\frac{\partial R}{\partial \partial_{\rho\sigma} g_{\mu\nu}}\delta_{\rho\sigma} g_{\mu\nu}\delta^{\rho}_{\lambda} \\
&= \frac{f' g^{\alpha\gamma} g^{\delta\eta}}{2}[\delta^{\mu}_{\delta}\delta^{\nu}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\alpha} - \delta^{\mu}_{\alpha}\delta^{\nu}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta} \\
&- \delta^{\mu}_{\delta}\delta^{\nu}_{\gamma}\delta^{\rho}_{\eta}\delta^{\sigma}_{\alpha} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\gamma}\delta^{\rho}_{\delta}\delta^{\sigma}_{\eta}]\partial_{\rho\sigma} g_{\mu\nu}\delta^{\rho}_{\lambda} \\
&= \frac{f' g^{\alpha\gamma} g^{\delta\eta}}{2}[\partial_{\gamma\alpha} g_{\delta\eta} - \partial_{\gamma\delta} g_{\alpha\eta} \\
&- \partial_{\eta\alpha} g_{\delta\gamma} + \partial_{\delta\eta} g_{\alpha\gamma}]\delta^{\rho}_{\lambda} \\
&= \sqrt{g}\mathcal{L}' R\delta^{\rho}_{\lambda}.
\end{aligned} \tag{27}$$

Combining equation (27),(26), and equation (24), the energy-momentum tensor will have the form

$$T^{\rho}_{\lambda} = 2\mathcal{L}' R\delta^{\rho}_{\lambda} - 3\mathcal{L}\delta^{\rho}_{\lambda} + \mathcal{L}''(g^{\rho\sigma}R_{;\lambda;\sigma} - 4\delta^{\rho}_{\lambda}\Box^2 R)$$
$$+\mathcal{L}'''(g^{\rho\sigma}R_{;\lambda}R_{;\sigma} - 4\delta^{\rho}_{\lambda}g^{\rho\sigma}R_{;\rho}R_{;\sigma})$$

If we multiply both sides by $g_{\gamma\rho}$ the resulting equations become

$$T_{\gamma\lambda} = 2g_{\gamma\lambda}R\mathcal{L}' - 3g_{\gamma\lambda}\mathcal{L} + \mathcal{L}''(R_{;\gamma;\lambda} - 4g_{\gamma\lambda}\Box^2 R) + \mathcal{L}'''(R_{;\gamma}R_{;\lambda} - 4g_{\gamma\lambda}g^{\rho\sigma}R_{;\rho}R_{;\sigma})$$

then interchanging $\mu\nu$ with $\gamma\lambda$ yields

$$T_{\mu\nu} = 2g_{\mu\nu}R\mathcal{L}' - 3g_{\mu\nu}\mathcal{L} + \mathcal{L}''(R_{;\mu;\nu} - 4g_{\mu\nu}\Box^2 R)$$
$$+\mathcal{L}'''(R_{;\mu}R_{;\nu} - 4g_{\mu\nu}g^{\rho\sigma}R_{;\rho}R_{;\sigma}) \qquad (28)$$

This expression can be simplified by using the GFE []

$$\mathcal{L}'''(R_{;\mu}R_{;\nu} - g_{\mu\nu}g^{\rho\sigma}R_{;\rho}R_{;\sigma}) + \mathcal{L}''(R_{;\mu;\nu} - g_{\mu\nu}\Box^2 R) + \mathcal{L}'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = 0$$

contracting this equation one gets

$$3\mathcal{L}'''R_{;\rho}R^{\rho}_{;} + 3\mathcal{L}''\Box^2 R = \mathcal{L}'R - 2\mathcal{L}$$

The term $g_{\mu\nu}\mathcal{L}$ is then given by

$$g_{\mu\nu}\mathcal{L} = 2\mathcal{L}'R_{\mu\nu} + 2\mathcal{L}''(R_{;\mu;\nu} - g_{\mu\nu}\Box^2 R) + 2\mathcal{L}'''(R_{;\mu}R_{;\nu} - g_{\mu\nu}g^{\rho\sigma}R_{;\rho}R_{;\sigma}).$$

Using the contracted equation in (28) one obtains

$$\begin{aligned} T_{\mu\nu} &= 2g_{\mu\nu}R\mathcal{L}' - 3g_{\mu\nu}\mathcal{L} - 3\mathcal{L}''g_{\mu\nu}\Box^2 R - 3\mathcal{L}'''g_{\mu\nu}R_{;\rho}R_{;}{}^{\rho} \\ &+ \mathcal{L}''(R_{;\mu;\nu} - g_{\mu\nu}\Box^2 R) + \mathcal{L}'''(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\rho}R_{;}{}^{\rho}) \\ &= g_{\mu\nu}(\mathcal{L}'R - \mathcal{L}) + \mathcal{L}''(R_{;\mu;\nu} - g_{\mu\nu}\Box^2 R) \\ &+ \mathcal{L}'''(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\rho}R^{\rho}_{;}) \end{aligned} \qquad (29)$$

Substituting for $g_{\mu\nu}\mathcal{L}$ in equation (29) yields the following equivalent expression

$$\begin{aligned} T_{\mu\nu} &= 2\mathcal{L}'(\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}) + \mathcal{L}''(g_{\mu\nu}\Box^2 R - R_{;\mu;\nu}) \\ &+ \mathcal{L}'''(g_{\mu\nu}R_{;\rho}R^{\rho}_{;} - R_{;\mu}R_{;\nu}) \end{aligned} \qquad (30)$$

Equations (29,30) give a Lagrangian-dependent expression for the energy-momentum tensor for the gravitational field . By choosing

$$\mathcal{L} = -\alpha R^2 + \beta R \ , \qquad (31)$$

being the simplest non-linear Lagrangian density which represents the most rational form, (29) yields

$$T_{\mu\nu} = -g_{\mu\nu}(\alpha R^2) - 2\alpha(R_{;\mu;\nu} - g_{\mu\nu}\Box^2 R). \qquad (32)$$

In this expression the terms with the coefficient α are assumed to represent the contribution of the strong gravitational field which in the case of weak gravity is set to zero. Therefore for the weak field, $\mathcal{L} = \mathcal{R}$ equation (29) gives $T_{\mu\nu} = 0$. Also in the case of free space equations (30) and (31) reduce to

$$T_{\mu\nu} = 2(\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}) = 0 \tag{33}$$

The vanishing of $T_{\mu\nu}$ can also be obtained from contraction of GFE for linear Lagrangian to get $R = 8\pi G T^{\lambda}_{\lambda} = 0$ outside the source, and for $\mathcal{L} = R$ equation (28) reduces to

$$T_{\mu\nu} = 2g_{\mu\nu}R - 3g_{\mu\nu}R = -g_{\mu\nu}R = 0$$

It is also noticed that the linear terms with the coefficient β canceled out in (32) and only the non-linear terms contribute to the energy-momentum tensor, thus when linear term only exists in the Lagrangian $\alpha = 0$. In this case equation (32) gives $T_{\mu\nu} = 0$ this may explain why GR which is derivable from $\mathcal{L} = R$ predicts zero energy in contrast with other field theories [8,9,10].

III. THE HAMILTONIAN IN THE NEWTONIAN LIMIT

To see how does the expression for the energy-momentum tensor satisfy the Newtonian limit, we shall consider the Hamiltonian density $\mathcal{H}$ given by equations (21) and (28) in the form

$$\begin{aligned}\mathcal{H} = -T^{0}_{\ 0} \;&=\; \mathcal{L} + \left(\partial_\sigma \frac{\partial \mathcal{L}}{\partial\partial_{0\sigma} g_{\mu\nu}}\partial_0 g_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial\partial_0 g_{\mu\nu}}\right)\partial_0 g_{\mu\nu} \\ &-\; \frac{\partial \mathcal{L}}{\partial\partial_{0\sigma} g_{\mu\nu}}\partial_{0\sigma} g_{\mu\nu}\end{aligned} \tag{34}$$

or equivalently

$$\begin{aligned}\mathcal{H} = -T^0_0 = 3\mathcal{L} - 2\mathcal{L}'R + 4\mathcal{L}''\Box^2 R - \mathcal{L}'' g^{tt} R_{;t;t} \\ +\mathcal{L}'''(4g^{\rho\sigma}R_{;\rho}R_{;\sigma} - g^{tt}R^2_{;t})\end{aligned} \tag{35}$$

It is clear that T^0_0 has the dimension of an energy density. Further the conservation of $T^{\rho}_{\ \lambda}$ is given from equation (12) as

$$\frac{\partial T^{\rho}_{\ \lambda}}{\partial x^\rho} = 0,$$

for $\lambda = 0$

$$\frac{\partial T^{\rho}_{\ 0}}{\partial x^\rho} = \frac{\partial T^{0}_{\ 0}}{\partial x^0} + \frac{\partial T^{1}_{\ 0}}{\partial x^1} + \frac{\partial T^{2}_{\ 0}}{\partial x^2} + \frac{\partial T^{3}_{\ 0}}{\partial x^3} \tag{36}$$

where

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ \text{and}\ x^3 = z$$

If we set

$$S^1 = -cT^{1}_{\ 0},\ S^2 = -cT^{2}_{\ 0},\ S^3 = -cT^{3}_{\ 0} \tag{37}$$

Then equation (12) reads

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla.\mathbf{S} = 0. \tag{38}$$

This equation represents the equation of continuity [18], where $\mathbf{S}$ is the energy flux density and the momentum of the system is given by

$$p^{\alpha} = -\frac{T^{\alpha}_{0}}{c}. \tag{39}$$

We conclude that T^{ρ}_{0} represents the energy-momentum four-vector and that the dimensional analysis as well as the continuity equation indicates that the four-vector T^{ρ}_{0} in equation (28) represents the components of the energy-momentum tensor

A. The Energy of a Static Isotropic Gravitational Field

Let us look for the form of the Hamiltonian density $\mathcal{H}$ in a static gravitational field of a spherical body of mass M. The field generated by this body is described by static isotropic metric which has the following form

$$g_{rr} = A(r),\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta,\ g_{tt} = -B(r)$$

Considering the case of a particle moving in static weak field, the geodesic equation yields[]

$$B(r) = 1 + 2\phi,\ \text{and}\ \phi = -\frac{MG}{r} \tag{40}$$

with ϕ the Newtonian potential. The scalar curvature R is thus given by

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} + g^{tt}R_{tt} \\ &= \frac{\ddot{B}}{2AB} - \frac{\dot{B}}{4AB}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) - \frac{\dot{A}}{rA^2} - \frac{2}{r^2} \\ &+ \frac{1}{rA}\left(\frac{-\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{2}{r^2A} \\ &+ \frac{\ddot{B}}{2AB} - \frac{\dot{B}}{4AB}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{\dot{B}}{rAB} \\ &= \frac{\ddot{B}}{AB} + \frac{2\dot{B}}{rAB} - \frac{\dot{B}}{2AB}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \\ &- \frac{2\dot{A}}{rA^2} - \frac{2}{r^2} + \frac{2}{r^2A} \end{aligned} \tag{41}$$

The function $A(r)$ in the expression of R can be found from equation the static solution of the GFE to be []

$$A(r) = \frac{[-r\frac{d}{dr}(\frac{\dot{B}r}{B})]}{\frac{\dot{B}r}{B} - r^2(2-\frac{\dot{B}r}{B})\xi} = -r\frac{[\frac{r}{B}\nabla^2 B - \frac{\dot{B}}{B}(1+\frac{\dot{B}r}{B})]}{[\frac{\dot{B}r}{B} - r^2(2-\frac{\dot{B}r}{B})\xi]} \qquad (42)$$

where

$$\xi = \frac{1}{3}\left(R - \frac{\mathcal{L}}{\mathcal{L}'}\right). \qquad (43)$$

when one is outside the source

$$\nabla^2 B = \ddot{B} + \frac{2}{r}\dot{B} = 0.$$

In this case $A(r)$ becomes

$$A(r) = \frac{[\frac{\dot{B}r}{B}(1+\frac{\dot{B}r}{B})]}{[\frac{\dot{B}r}{B} - r^2(2-\frac{\dot{B}r}{B})\xi]}. \qquad (44)$$

The most simple non linear Lagrangian for static isotropic metric may be chosen to have the form

$$\mathcal{L} = -\alpha R^2 + \beta R + \gamma, \ \mathcal{L}' = -2\alpha R + \beta$$
$$\mathcal{L}'' = -2\alpha, \ \mathcal{L}''' = 0 \qquad (45)$$

as a result ξ takes the form

$$\xi = \frac{1}{3}\left(R - \frac{\mathcal{L}}{2\mathcal{L}'}\right) = \frac{1}{3}\left[R - \frac{(-\alpha R^2 + \beta R + \gamma)}{-4\alpha R + 2\beta}\right]. \qquad (46)$$

By assuming α as characteristic to the strong field then the terms proportional to α are negligible for the weak field, therefore

$$\xi = \frac{1}{6}R - \frac{\gamma}{6\beta}. \qquad (47)$$

Using equation (44) together with equation (47) and setting $k = \gamma/\beta$ yields

$$\frac{1}{r^2 A} = \frac{B}{r^2} - B\left(\frac{2B}{\dot{B}r} - 1\right)\left(\frac{R-k}{6}\right) \qquad (48)$$

on the other hand from (40) one gets

$$B + \dot{B}r = 1$$

To simplify the calculation it is convenient to define a function z by

$$z = \frac{1}{r^2 A}, \ \dot{z} = -\frac{\dot{A}}{r^2 A^2} - \frac{2}{r^3 A}, \ -\frac{\dot{A}}{A^2} = r^2 \dot{z} + 2rz. \tag{49}$$

Substituting this in equation (41) and taking into account that at a sufficiently large distance from the gravitating source

$$\frac{\ddot{B}}{AB} + \frac{2\dot{B}}{rAB} = \frac{1}{AB}\nabla^2 B = 0,$$

this yields

$$R = \left(\frac{\dot{B}r^2}{2B} + 2r\right)\dot{z} + \left(\frac{\dot{B}r}{B} - \frac{\dot{B}^2 r^2}{2B^2} + 6\right)z - \frac{2}{r^2} \tag{50}$$

Then substituting the above expressions in equation (48) one gets

$$z = \frac{B}{r^2} + \frac{Bk}{6}\left(\frac{2B}{\dot{B}r} - 1\right)$$

$$-\frac{B}{6}\left(\frac{2B}{\dot{B}r} - 1\right)\left[\left(\frac{\dot{B}r^2}{2B} + 2r\right)\dot{z} + \left(\frac{\dot{B}r}{B} - \frac{\dot{B}^2 r^2}{2B^2} + 6\right)z - \frac{2}{r^2}\right].$$

Rearranging this equation we have

$$a(r)\dot{z}(r) + b(r)z(r) = c(r) \tag{51}$$

where

$$a(r) = \left(\frac{\dot{B}r^2}{2B} + 2r\right)\left(\frac{1}{r} - \frac{\dot{B}}{2B}\right),$$

$$b(r) = \left(\frac{1}{r} - \frac{\dot{B}}{2B}\right)\left(\frac{\dot{B}r}{B} - \frac{\dot{B}^2 r^2}{2B^2} + 6\right) + \frac{3\dot{B}}{B^2}$$

and

$$c(r) = \left(\frac{2\dot{B}}{r^2 B} + \frac{2}{r^3}\right) + k\left(\frac{1}{r} - \frac{\dot{B}}{2B}\right). \tag{52}$$

Multiplying both sides of (51) by the integrating factor μ yields

$$\mu\dot{z} + \mu\frac{b}{a}z = \mu\frac{c}{a}. \tag{53}$$

If μ is selected such that

$$\frac{d\mu z}{dr} = \mu\dot{z} + \dot{\mu}z = \mu\dot{z} + \mu\frac{b}{a}z = \frac{\mu c}{a} \tag{54}$$

Then $\dot{\mu} = \mu b/a$ and hence

$$\ln\mu = c_1 + \int \frac{b}{a}dr$$

$$\mu = c_2 e^{\int \frac{b}{a} dr} \tag{55}$$

$$\frac{d\mu z}{dr} = \frac{\mu c}{a} \text{ , and } z = \frac{c_3}{\mu} + \frac{1}{\mu}\int \frac{\mu c}{a} dr \tag{56}$$

where c_1, c_2, c_3 are integration constants.

To obtain μ let us simplify the ratio $\frac{b}{a}$ from (52) as follows

$$\frac{b}{a} = \frac{-\frac{\dot{B}}{B}(\frac{\dot{B}r^2}{2B} + 2r)}{\frac{\dot{B}r^2}{2B} + 2r} + \frac{3\frac{\dot{B}}{B}r + 6}{\frac{\dot{B}r^2}{2B} + 2r}$$

$$+12\frac{\dot{B}}{(2\dot{B} - \dot{B}r)(\dot{B}r + 4B)}$$

i.e.

$$\frac{b}{a} = -\frac{\dot{B}}{B} + \frac{6(\dot{B}r + 2B)}{r(\dot{B}r + 4B)} + 12\frac{\dot{B}}{(2\dot{B} - \dot{B}r)(\dot{B}r + 4B)} \tag{57}$$

On the other hand

$$B = 1 - \frac{2MG}{r}, \ \dot{B} = \frac{2MG}{r^2} \ ,$$

$$2B + \dot{B}r = \frac{2}{r}(r - MG), \ 4B + \dot{B}r = \frac{2}{r}(r - 3MG)$$

$$\text{and } \ 2B - \dot{B}r = \frac{2}{r}(r - 3MG),$$

Hence by substitution we get

$$\frac{b}{a} = -\frac{\dot{B}}{B} + \frac{6(r - MG)}{r(2r - 3MG)} + \frac{6MG}{(r - 3MG)(2r - 3MG)}$$

The second and third terms can be factorized to get

$$\frac{6(r - MG)}{r(2r - 3MG)} = \frac{2}{r} + \frac{2}{(2r - 3MG)}$$

$$\frac{6MG}{(r - 3MG)(2r - 3MG)} = \frac{-4}{(2r - 3MG)} + \frac{2}{(r - 3MG)}$$

$$\frac{b}{a} = -\frac{\dot{B}}{B} + \frac{2}{r} - \frac{2}{(2r - 3MG)} + \frac{2}{(r - 3MG)}$$

Hence

$$\int \frac{b}{a} dr = -\int d\ln B + 2\int d\ln r - \int d\ln(2r - 3MG) + 2\int d\ln(r - 3MG)$$

or

$$\int \frac{b}{a} dr = \ln \frac{r^2(r - 3MG)^2}{B(2r - 3MG)} \tag{58}$$

Substituting (58) in (55) yields

$$\mu(r) = \frac{c_2 r^2 (r-3MG)^2}{B(2r-3MG)} \tag{59}$$

Similarly c/a can be simplified using the following set of equations

$$\frac{c}{a} = \frac{8B(\dot{B}r+B)}{r^3(2B-\dot{B}r)(\dot{B}r+4B)} + \frac{2kB}{r(\dot{B}r+4B)}$$

and

$$\frac{c}{a} = \frac{2B}{r(r-3MG)(2r-3MG)} + \frac{kB}{(2r-3MG)} \tag{60}$$

Multiplying both sides by $\mu(r)$ yields

$$\frac{\mu c}{a} = \frac{2c_2 r(r-3MG)}{(2r-3MG)^2} + \frac{c_2 k r^2 (r-3MG)^2}{(2r-3MG)^2}.$$

In order to integrate this expression we introduce the following substitutions

$$\eta \equiv 2r - 3MG, \; r = \frac{1}{2}(\eta + 3MG), \; dr = \frac{1}{2} d\eta \tag{61}$$

, and therefore,

$$\int \frac{\mu c}{a} dr = \frac{c_2}{4} \int \frac{(\eta+3MG)(\eta-3MG)}{\eta^2} d\eta$$
$$+ \frac{c_2 k}{32} \int \frac{(\eta+3MG)^2(\eta-3MG)^2}{\eta^2} d\eta$$
$$= \frac{c_2}{4} \int d\eta - \frac{9c_2 M^2 G^2}{4} \int \frac{d\eta}{\eta^2} + \frac{c_2 k}{32} \int \frac{(\eta^4 - 18M^2G^2\eta^2 + 81M^4G^4)}{\eta^2} d\eta$$
$$= \frac{c_2}{4}\eta + \frac{9c_2 M^2 G^2}{4\eta} + \frac{c_2 k}{96}\eta^3 - \frac{9c_2 M^2 G^2 k}{16}\eta - \frac{81 c_2 M^4 G^4 k}{32\eta}$$

i.e.

$$\int \frac{\mu c}{a} dr = \frac{c_2 k (2r-3MG)^3}{96}$$
$$+ \frac{c_2(4 - 9M^2G^2k)(2r-3MG)}{16} + \frac{9M^2G^2 c_2 (8 - 9M^2G^2k)}{32(2r-3MG)} \tag{62}$$

Combining equations (56),(59) and (62) yields

$$\begin{aligned} z &= \frac{B(2r-3MG)}{r^2(r-3MG)^2}\Big[\frac{c_3}{c_2} + \frac{k(2r-3MG)^3}{96} \\ &+ \frac{(4-9M^2G^2k)(2r-3MG)}{16} \\ &+ \frac{9M^2G^2(8-9M^2G^2k)}{32(2r-3MG)}\Big] \end{aligned} \tag{63}$$

Dividing the numerator and the denominator by r^4 we get

$$z = \frac{B(2-\frac{3MG}{r})}{(1-\frac{3MG}{r})^2}\left[\frac{c_3}{c_2}\frac{1}{r^3}+\frac{k}{96}\left(2-\frac{3MG}{r}\right)^3\right]$$
$$+\left[\frac{(4-9M^2G^2k)}{16}\left(\frac{2}{r^2}-\frac{3MG}{r^3}\right)+\frac{9M^2G^2(8-9M^2G^2k)}{32r^3(2r-3MG)}\right]$$

but

$$\left(2-\frac{3MG}{r}\right)^4 = \left(4-\frac{12MG}{r}+\frac{9M^2G^2}{r^2}\right)\left(4-\frac{12MG}{r}+\frac{9M^2G^2}{r^2}\right)$$
$$= 16-\frac{96MG}{r}+\frac{216M^2G^2}{r^2}-\frac{216M^3G^3}{r^3}+\frac{81M^4G^4}{r^4}.$$

and when r is sufficiently large the terms $\frac{1}{r^3}$, $\frac{1}{r^4}$ can be neglected comparec $1/r$ and $1/r^2$ in the numerator and this gives

$$z = \frac{B}{(1-\frac{3MG}{r})^2}\left[\frac{k}{96}\left(16-\frac{96MG}{r}+\frac{216M^2G^2}{r^2}\right)+\right.$$
$$\left.\frac{(4-9M^2G^2k)}{16}\left(2-\frac{3MG}{r}\right)\frac{2}{r^2}\right]$$
$$= \frac{B}{(1-\frac{3MG}{r})^2}\left[\frac{k}{96}\left(16-\frac{96MG}{r}+\frac{216M^2G^2}{r^2}\right)+\frac{1}{r^2}-\frac{9M^2G^2k}{4r^2}\right]$$

or

$$\frac{z}{B} = \frac{(\frac{k}{6}-\frac{kMG}{r}+\frac{9M^2G^2k}{r^2}+\frac{1}{r^2}-\frac{9M^2G^2k}{4r^2})}{(1-\frac{3MG}{r})^2}$$

Far away from the source then the denominator becomes

$$\left(1-\frac{3MG}{r}\right)^2 \approx 1,$$

and our expression reduces to

$$z = (\frac{k}{6}-\frac{kMG}{r}+\frac{1}{r^2})B \tag{64}$$

Using the definition (49) one gets

$$\frac{1}{r^2AB} = \frac{k}{6}-\frac{kMG}{r}+\frac{1}{r^2} \tag{65}$$

$$\frac{1}{AB} = \frac{kr^2}{6}-kMGr+1. \tag{66}$$

When $k = 0$, $1/AB = 1$, this represents Schwarzchild solution [19]. By further approximation in equation (65) and dropping terms of the order $1/r$, $1/r^2$ on the right hand side we get

$$\frac{1}{r^2 A} = \frac{kB}{6}. \tag{67}$$

Equation (66) can also be used to find

$$\frac{d}{dr}\left(\frac{1}{A}\right) = -\frac{\dot{A}}{A^2} = \dot{B}\left(\frac{kr^2}{6} - kMGr + 1\right) + \left(\frac{kr}{3} - kMG\right) B \tag{68}$$

Equation (66) together with equations (67) and (68) and the identity

$$\ddot{B} + \frac{2\dot{B}}{r} = \nabla^2 B = 2\nabla^2\phi = 8\pi G\rho \tag{69}$$

can be used in equation (41)to get

$$\begin{aligned} R &= \frac{8\pi G\rho}{AB} + \frac{\dot{B}}{2B}\left(-\frac{\dot{A}}{A^2}\right) - \frac{\dot{B}^2}{2B}\left(\frac{1}{AB}\right) + \frac{2}{r}\left(-\frac{\dot{A}}{A^2}\right) - \frac{2}{r^2} + \frac{2}{r^2 A} \\ &= 8\pi G\rho\left(1 + \frac{kr^2}{6} - kMGr\right) + \frac{\dot{B}^2}{2B}\left(\frac{kr^2}{6} - kMGr + 1\right) \\ &+ \frac{\dot{B}}{2}\left(\frac{kr}{3} - kMG\right) - \frac{\dot{B}^2}{2B}\left(\frac{kr^2}{6} - kMGr + 1\right) \\ &+ 2\dot{B}\left(\frac{kr}{6} - kMG + \frac{1}{r}\right) + 2B\left(\frac{k}{3} - \frac{kMG}{r}\right) \\ &- \frac{2}{r^2} + \frac{k}{3}B. \end{aligned} \tag{70}$$

By setting

$$\mu = \frac{kr^2}{6} - kMGr \tag{71}$$

and using

$$1/AB = (1 + \mu) \quad \text{and} \quad B = 1 - 2MG/r = 1 + 2\phi,$$

i.e.

$$\dot{B} = \frac{2MG}{r^2} = 2\nabla\phi$$

and

$$\dot{B}kr = k\frac{2MG}{r} = k(1 - B),$$

and also

$$\begin{aligned} -\frac{2kMG}{r}B &= k\left(-\frac{2MG}{r}\right) + \frac{4M^2G^2k}{r^2} \\ &= kB - k + \frac{4M^2G^2k}{r^2} \end{aligned} \tag{72}$$

we then obtain

$$R = 8\pi G\rho(1+\mu) + \frac{k\dot{B}r}{2} + \frac{2\dot{B}}{r} - \frac{5}{2}kMG\dot{B} -$$
$$\frac{2MG}{r}kB - \frac{2}{r^2} + kB$$
$$= 8\pi G\rho(1+\mu) + \frac{k}{2} - \frac{k}{2}B + \frac{2\dot{B}}{r} - \frac{5}{2}kMG\dot{B} + kB -$$
$$k + \frac{4M^2G^2k}{r^2} - \frac{2}{r^2} + kB.$$

Neglecting terms of order $\frac{1}{r^2}$, $\frac{1}{r^3}$, ..., the scalar curvature becomes simplified to read

$$R = 8\pi G\rho(1+\mu) - \frac{k}{2} + \frac{3}{2}kB$$
$$= 8\pi G\rho(1+\mu) + k + 3k\phi. \tag{73}$$

Differentiating both sides with respect to r and assuming ρ to be constant one gets

$$\dot{R} = 8\pi G\dot{\mu}\rho + 3k\nabla\phi \tag{74}$$

and using the fact that $k = \gamma/\beta$ in equation (48) and setting $\beta = 1/16\pi G$ together with equations (73) and (74) it follows that

$$\frac{1}{3}\beta R = \frac{(1+\mu)\rho}{6} + \frac{\gamma}{3} + \gamma\phi \tag{75}$$

and

$$\frac{\alpha\dot{B}}{AB}\dot{R} = 2\alpha(1+\mu)(\nabla\phi)(8\pi G\dot{\mu}\rho + 3k\nabla\phi)$$
$$= \frac{6\alpha\gamma}{\beta}(\nabla\phi)^2 + \frac{6\alpha\gamma\mu}{\beta}(\nabla\phi)^2 + 16\pi G\alpha(1+\mu)\dot{\mu}\rho\nabla\phi. \tag{76}$$

Further, the Hamiltonian $\mathcal{H}$ can be found from equations (35) and(45) to be

$$\mathcal{H} = 3\mathcal{L} - 2\mathcal{L}'R + 4\mathcal{L}''\Box^2 R - \mathcal{L}''g^{tt}R_{;t;t} + \mathcal{L}'''(4g^{\rho\sigma}R_{;\rho}R_{;\sigma} - g^{tt}R_{;t}^2)$$

If we set

$$\mathcal{L} = -\alpha R^2 + \beta R + \gamma,$$

that means

$$\mathcal{L}' = -2\alpha R + \beta,\ \mathcal{L}'' = -2\alpha\ \text{, and }\ \mathcal{L}''' = 0$$

therefore

$$\mathcal{H} = \alpha R^2 + \beta R + 3\gamma - 8\alpha\Box^2 R + \frac{\alpha\dot{B}}{AB}\dot{R} \tag{77}$$

This expression can be simplified by using the contracted form of the GFE (see the equation just after (28) i.e.

$$3\mathcal{L}'''R_{;\rho}R^{;\rho} + 3\mathcal{L}''\Box^2 R = \mathcal{L}'R - 2\mathcal{L}$$

which yields

$$8\alpha\Box^2 R = \frac{4}{3}\beta R + \frac{8}{3}\gamma,$$

The Hamiltonian will then be given by

$$\mathcal{H} = \alpha R^2 - \frac{1}{3}\beta R + \frac{1}{3}\gamma + \frac{\alpha \dot{B}}{AB}\dot{R} \tag{78}$$

In a weak field limit R^2 can be neglected compared to R and the Hamiltonian becomes

$$\mathcal{H} = -\frac{1}{3}\beta R + \frac{1}{3}\gamma + \frac{\alpha \dot{B}}{AB}\dot{R} \tag{79}$$

To obtain $\mathcal{H}$ in the static limit we make use of equations (75) and (76) in equation (79)

$$\mathcal{H} = -\frac{(1+\mu)\rho}{6} - \frac{\gamma}{3} - \gamma\phi + \frac{\gamma}{3} + \frac{6\alpha\gamma}{\beta}(\nabla\phi)^2$$

$$+\frac{6\alpha\gamma}{\beta}\mu(\nabla\phi)^2 + 16\pi G\alpha(1+\mu)\dot{\mu}\rho\nabla\phi$$

If we substitute

$$\gamma = c\rho, \ \alpha = \frac{n\beta^2}{\gamma}, \ \text{and} \ \beta = \frac{1}{16\pi G} \tag{80}$$

where c is a certain constant, then we will have

$$\mathcal{H} = \frac{-\rho}{6} - c\rho\phi + \frac{3n}{8\pi G}(\nabla\phi)^2$$

$$+\frac{3n\mu}{8\pi G}(\nabla\phi)^2 + 16\pi G\alpha(1+\mu)\dot{\mu}\rho\nabla\phi$$

by setting

$$c = \frac{1}{6}, \ \text{and}, \ n = \frac{1}{18}$$

the expression of the Hamiltonian becomes

$$\mathcal{H} = -\frac{1}{6}\left(\rho + \rho\phi - \frac{(\nabla\phi)^2}{8\pi G}\right) + \frac{\mu(\nabla\phi)^2}{48\pi G} + 16\pi G\alpha(1+\mu)\dot{\mu}\rho\nabla\phi \tag{81}$$

The first term in this expression is proportional to the expression which represents the Hamiltonian in a Newtonian limit [20,21]. This indicates that our expression (35) for the Hamiltonian satisfies the principle of correspondence . It is important to note that when the Lagrangian $\mathcal{L}$ contains no quadratic terms i.e. $\alpha = 0$ and by (80) $n = 0$ then the expression for the gravitational energy disappears from this Hamiltonian. This indicates that the existence of the gravitational energy depends on the existence of the quadratic term in the Lagrangian or equivalently the gravitational energy exists only when the quadratic term αR^2 is included in the Lagrangian. This supports the afore-noted conclusion that the failure of GR to give a well defined expression for the energy of the gravitational field is due to its being derived from the linear Lagrangian .

B. The Quasi -Minkowskian Approximation

The Newtonian limit can also be obtained by using quasi-Minkowskian approximation [14], where the metric $g_{\mu\nu}$ approaches the Minkowskian metric $\eta_{\mu\nu}$ when we consider the field of a far away point from the source. In this case $g_{\mu\nu}$can be written as in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{82}$$

where $|h_{\mu\nu}| \ll 1$. It follows that the first order part of the Ricci tensor which is linear in $h_{\mu\nu}$ is given by

$$R^{(1)}_{\mu\kappa} = \frac{1}{2}(\frac{\partial^2 h^\lambda_\lambda}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 h^\lambda_\mu}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h^\lambda_\kappa}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x_\lambda}) \tag{83}$$

The second order part of the Ricci tensor $R^{(2)}_{\mu\kappa}$is given by

$$R^{(2)}_{\mu\kappa} = -\frac{1}{2}h^{\lambda\nu}[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}]$$
$$+\frac{1}{4}[2\frac{\partial h^\nu_\sigma}{\partial x^\nu} - \frac{\partial h^\nu_\nu}{\partial x^\sigma}][\frac{\partial h^\sigma_\mu}{\partial x^\kappa} + \frac{\partial h^\sigma_\kappa}{\partial x^\mu} - \frac{\partial h_{\mu\kappa}}{\partial x^\sigma}]$$
$$-\frac{1}{4}[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}][\frac{\partial h^\sigma_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda_\mu}{\partial x_\sigma}] \tag{84}$$

In the case of a weak and static field

$$h_{00} = -2\phi(r) \tag{85}$$

where ϕ is the Newtonian potential . Considering the time-time components of $h_{\mu\nu}$ corresponding to the Hamiltonian density which represent the zero-zero component of the energy-momentum tensor, then the first and the second part of the Ricci tensor yield

$$R^{(1)}_{00} = \frac{1}{2}(\frac{\partial^2 h^0_0}{\partial x^0 \partial x^0} - \frac{\partial^2 h^0_0}{\partial x^\lambda \partial x^0} - \frac{\partial^2 h^0_0}{\partial x^\lambda \partial x^0} + \frac{\partial^2 h_{00}}{\partial x^i \partial x_i})$$
$$R^{(1)}_{00} = \frac{1}{2}(\frac{\partial^2 h^0_0}{\eta_{ii}\partial x^{i2}}) = \frac{1}{2}\frac{\partial^2 h_{00}}{\partial x^{i2}} \tag{86}$$

where

$$\frac{\partial h_{00}}{\partial x^0} = 0, \ \frac{\partial h^0_0}{\partial x^0} = 0, \ \frac{\partial h^{00}}{\partial x^0} = 0$$

and

$$R^{(1)}_{ii} = \frac{1}{2}(\frac{\partial^2 h^0_0}{\partial x^i \partial x^i} - \frac{\partial^2 h^i_i}{\partial x^{i2}} - \frac{\partial^2 h^i_i}{\partial x^{i2}} + \frac{\partial^2 h_{ii}}{\partial x^\lambda \partial x_\lambda})$$
$$R^{(1)}_{ii} = \frac{1}{2}(\frac{\partial^2 h^0_0}{\partial x^{i2}}) = \frac{1}{2}\frac{\partial^2 \eta^{00} h_{00}}{\partial x^{i2}} = -\frac{1}{2}\frac{\partial^2 h_{00}}{\partial x^{i2}} \tag{87}$$

with

$$\eta^{00} = -1, \ \eta^{ii} = 1 \tag{88}$$

Hence the first order part will be given by

$$R^{(1)} = \eta^{00} R_{00}^{(1)} + \eta^{ii} R_{ii}^{(1)} = -\frac{1}{2}\frac{\partial^2 h_{00}}{\partial x^{i2}} - \frac{1}{2}\frac{\partial^2 h_{00}}{\partial x^{i2}}$$

and

$$R^{(1)} = -\nabla^2 h_{00} = 2\nabla^2\phi = 8\pi G\rho \qquad (89)$$

The components of the second order part are given by

$$R_{00}^{(2)} = -\frac{1}{2}h^{\lambda\nu}[\frac{\partial^2 h_{00}}{\partial x^\nu \partial x^\lambda}] + \frac{1}{4}[-\frac{\partial h_0^0}{\partial x^i}][-\frac{\partial h_{00}}{\partial x_i}]$$
$$-\frac{1}{4}[\frac{\partial h_{00}}{\partial x^i}][\frac{\partial h_0^0}{\partial x_i}] - \frac{1}{4}[\frac{\partial h_{00}}{\partial x^i}][\frac{\partial h_0^0}{\partial x_i}] \qquad (90)$$

i.e.

$$R_{00}^{(2)} = -\frac{1}{2}h^{00}[\frac{\partial^2 h_{00}}{\partial x^{02}}] - \frac{1}{2}h^{ii}[\frac{\partial^2 h_{00}}{\partial x^{i2}}] + \frac{1}{4}[\frac{\partial \eta^{00} h_{00}}{\partial x^i}][\frac{\partial h_{00}}{\partial \eta_{ii} x^i}]$$
$$-\frac{1}{4}[\frac{\partial h_{00}}{\partial x^i}][\frac{\partial \eta^{00} h_{00}}{\partial \eta_{ii} x^i}] - \frac{1}{4}[\frac{\partial h_{00}}{\partial x^i}][\frac{\partial \eta^{00} h_{00}}{\partial \eta_{ii} x^i}]$$
$$= -\frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})^2 + \frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})^2 + \frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})^2$$
$$= \frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})^2 = \frac{1}{4}(\nabla h_{00})^2 = (\nabla\phi)^2 \qquad (91)$$

and

$$R_{ii}^{(2)} = -\frac{1}{2}h^{00}(\frac{\partial^2 h_{00}}{\partial x^{i2}}) - \frac{1}{4}\frac{\partial h_{00}}{\partial x^i}(\frac{\partial h^{00}}{\partial x^i})$$
$$= -\frac{\eta^{00}\eta^{00}h_{00}}{2}(\frac{\partial^2 h_{00}}{\partial x^{i2}}) - \frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})(\frac{\partial \eta^{00}\eta^{00}h_{00}}{\partial x^i}) \qquad (92)$$
$$= -\frac{h_{00}}{2}\frac{\partial^2 h_{00}}{\partial x^{i2}} - \frac{1}{4}(\frac{\partial h_{00}}{\partial x^i})^2 \qquad (93)$$
$$= -2\phi\frac{\partial^2\phi}{\partial x^{i2}} - (\frac{\partial\phi}{\partial x^i})^2 \qquad (94)$$

Hence the second order part of R becomes

$$R^{(2)} = \eta^{00}R_{00}^{(2)} + \eta^{ii}R_{ii}^{(2)} = -(\nabla\phi)^2 - 2\phi\eta^{ii}\frac{\partial^2\phi}{\partial x^{i2}} - \eta^{ii}(\frac{\partial\phi}{\partial x^i})^2$$
$$= -(\nabla\phi)^2 - 2\phi(\nabla\phi)^2 - (\nabla\phi)^2$$
$$= -2(\nabla\phi)^2 - 8\pi G\rho\phi \qquad (95)$$

The scalar curvature R is then given by

$$R = R^{(1)} + R^{(2)} = 8\pi G\rho - 2(\nabla\phi)^2 - 8\pi G\rho\phi \qquad (96)$$

using the Hamiltonian in equation (79) with

$$\dot{R} = -4(\nabla\phi)(\nabla^2\phi) - 8\pi G\rho(\nabla\phi) \tag{97}$$

and

$$\dot{R} = -24\pi G\rho(\nabla\phi). \tag{98}$$

also as

$$g_{\mu\nu} \to \eta_{\mu\nu},\ A \to 1,\ \text{and}\ B \to 1,$$

then $\mathcal{H}$ becomes

$$\mathcal{H} = -\frac{1}{3}\beta R + \frac{1}{3}\gamma + 2\alpha(\nabla\phi)^2(-24\pi G\rho)$$

$$= -\frac{1}{48\pi G}(8\pi G\rho - 2(\nabla\phi)^2 - 8\pi G\rho\phi) + \frac{1}{3}\gamma - 48\pi G\alpha\rho(\nabla\phi)^2$$

Setting $\gamma = c\rho,\ \alpha = n\beta^2/\gamma$ we get

$$\mathcal{H} = -\frac{1}{6}[\rho - \rho\phi - \frac{2(\nabla\phi)^2}{8\pi G}] + \frac{1}{3}c\rho - \frac{3n}{16\pi Gc}(\nabla\phi)^2$$

By using natural units $c = 1$, one obtains

$$\mathcal{H} = \frac{1}{6}[\rho + \rho\phi + \frac{2(\nabla\phi)^2}{8\pi G}] - \frac{3n}{16\pi G}(\nabla\phi)^2$$

If we choose $n = 1/3$ then

$$\mathcal{H} = \frac{1}{6}[\rho + \rho\phi - \frac{(\nabla\phi)^2}{8\pi G}] \tag{99}$$

This agrees with the expression of the Hamiltonian of matter in the presence of the gravity field which again confirms our belief that the absence of a well defined expression for the gravitational energy in GR is caused by its being based on a linear form of the Lagrangian. And thus this conclusion throws light on one of the limitations of Einstein's model of gravitation.

IV. CONCLUSION

In section (2) a useful expression for the energy-momentum tensor of the gravitational field is obtained, which is given by equations (28), (29) and (30). It was shown that the energy momentum tensor vanishes when the Lagrangian is linear and this may explain why GR does not possess a non-zero expression for the gravitational energy outside the source. In the weak field limit, when static isotropic metric is considered, the Hamiltonian consists of matter energy, gravity energy and interaction energy. The same result is obtained by using quasi-Minkowskian metric, thus the energy momentum satisfies the Newtonian limit, i.e in the weak field limit it gives Newtonian energy of the matter and gravitational field.

REFERENCES

1. *L. Janossy, Theory of Relativity based on Physical Reality (Akad'emiai Kaido', Budapest, 1971), chap.11*
2. *A.A.Logunov, Gravitation and elementary Particle Physics (Mir Publishers, Moscow, 1983), chap.1*
3. *Szeidl, G.: Boundary integral equations for plane problems in terms of stress functions of order one, Journal of Computational and Applied Mechanics, 2(2), (2001), 237-261.*
4. *Carlson D. E.: On G¨unther's stress functions for couple stresses, Quart. Appl. Math., 25, (1967), 139-146.*