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NURBS FINITE ELEMENT APPROACH FOR ONE DIMENSIONAL BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper, an attempt is made to use the Non Uniform Rational B-Spline (NURBS) basis functions as the shape functions in the finite element method. These basis functions are employed in Collocation Method approximation for the spatial descretization. It uses recursive formula of NURBS basis functions for solving second order differential equations with Neumann's boundary conditions. A test case is considered to study the efficiency of this method. When the number of nodes increased the stability and efficiency is improved. The result obtained by present method is compared and found to be in good agreement with analytical solution and finite element method.

Keywords: *NURBS,B-Splines, Isogeometric Method, Collocation Method, Cylindrical Fi*

I. INTRODUCTION

Mathematical models in the form of differential and partial differential equations are used to represent various engineering problems in the fields, such as Structural mechanics, Solid mechanics, Fluid flow, Heat transfer, Vibration analyses, Contact mechanics etc. The solutions to these mathematical models can be Exact, Analytical or Approximate depending on the nature of these equations. When the Exact solution is not possible, numerical methods are needed to obtain approximate solutions. Many numerical techniques are evolved and has been used increasingly in last few years. Those numerical techniques include Finite Difference Method [R.K.Panday *et al*, 2004], B-spline Collocation Method [Moshen.A *et al*, 2008], Predictor and Corrector Method [Abdalkaleg Hamad *et al*, 2014], Finite Element Method [Ch.Sridhar Reddy *et al*, 2014], and many more.

A non-uniform knot vector for a particular weight vector is used to obtain the second and third degree NURBS basis functions. For the spatial descretization, Collocation Method approximation is employed.

In this paper the recursive formulation of B-spline and NURBS basis functions [Hughes, T.J.R *et al*.2005, David F. Rogers *et al*, 2002] are discussed initially then the NURBS collocation method is discussed and formulated.

The effectiveness and accuracy of this method is tested using the governing equation of one dimensional boundary value problem.

Considering second order linear differential equations with variable coefficients

$$\frac{d^2U}{dx^2} + k_1 P(x)\frac{dU}{dx} + k_2 Q(x)U = F(x), \qquad a \le x \le b \qquad (1)$$

With the boundary conditions $U(a)=d1$, $U(b)=d2$. Where a,b,d_1,d_2, $k_1$and $k_2$are variables, $P(x)$, $Q(x)$ and $F(x)$ are functions of x.

Let the approximation solution be

$$U^h(x) = \sum_{i=-2}^{n-1} C_i R_{i,p}(x) \qquad (2)$$

Where C_i are constants to be determined and $R_{i,p}(x)$ are NURBS basis functions.

$U^h(x)$ is the approximate global solution to the exact solution U(x) of the considered second order singular differential equation(1).

II. A BRIEF INTRODUCTION TO B-SPLINES/NURBS

B-Splines

A spline is the mathematical representation of real world geometries. Schoenberg[David F. Rogers *et al*,2002] was given first reference to the word B-spline and described it as a smooth piecewise polynomial curve. From mathematical point of view, a curve generated by using the vertices of a defining polygon and the curve is dependent on some interpolation scheme between the curve and polygon. This scheme is provided by the choice of B-spline basis functions. B-spline basis is generally has non global behaviour due to the property that each vertex of B-spline B_i is associated with a unique basis function.

III. COLLOCATION METHOD

Collocation method is used widely in approximation methods particularly solving differential equations. The collocation method together with NURBS (Non-Uniform Rational Basis Spline) approximations represents an economical alternative since it only requires the evaluation of the unknown parameters at the grid points or nodes or collocation points. In normal collocation method we use polynomials whereas in NURBS collocation method we use NURBS basis functions. The selection of nodes or collocation points is arbitrary. The basis function vanishes at the boundary values. The success of this Collocation method is dependent on the choice of basis. The main aim is to analyse the efficiency of the NURBS based collocation method for such problems with sufficient accuracy.

Formulation of NURBS Collocation Method:

As mentioned earlier NURBS functions are used as basis in collocation method whereas the base functions which are used in the normal collocation method are polynomials, vanishes at the nodes. Let [*a, b*] be the domain of the governing differential equation and is partitioned as $X=\{a=x_1, x_2, \ldots\ x_{n-1}, x_n=b\}$with non-uniform values between [a b] with *n* sub domains. for a particular homogeneous coordinate(weights) i.e., h_i's. The x_i's are known as nodes, these nodes are treated as knots in collocation NURBS method, where NURBS basis functions are defined and those nodes are used to make the residue equal zero to determine unknowns C_i's in equation (2). Extra knot values are taken into consideration both sides of the domain of problem when evaluating the second degree NURBS basis functions at the nodes. These extra knots are taken to satisfy the partition of unity property and to get accurate NURBS basis functions.

First derivative of approximation function (2) is

$$\frac{dU^h(x)}{dx} = \sum_{i=-2}^{n-1} C_i R'_{i,p}(x) \tag{3}$$

Second derivative of approximation function (2) is

$$\frac{d^2U^h}{dx^2} = \sum_{i=-2}^{n-1} C_i R''_{i,p}(x) \tag{4}$$

Substituting, the approximate solution (2) in (1) we have,

$$\frac{d^2U^h}{dx^2} + k_1 P(x)\frac{dU^h}{dx} + k_2 Q(x)U^h = F(x) \tag{5}$$

Substituting the Approximation function and its derivatives (2),(3) and (4) in the equation (5), we have

$$\sum_{i=-2}^{n-1} C_i R''_{i,p}(x) + k_1 P(x)\sum_{i=-2}^{n-1} C_i R'_{i,p}(x) + k_2 Q(x)\sum_{i=-2}^{n-1} C_i R_{i,p}(x) = F(x) \tag{6}$$

Expanding the equation (6)

$$[C_{-2}R''_{-2,p}(x)+C_{-1}R''_{-1,p}(x)+C_{1}R''_{1,p}(x)+.......+C_{n-1}R''_{n-1,p}(x)]+$$
$$k_1P(x)[C_{-2}R'_{-2,p}(x)+C_{-1}R'_{-1,p}(x)+C_{1}R'_{1,p}(x)+.......+C_{n-1}R'_{n-1,p}(x)]+$$
$$k_2Q(x)[C_{-2}R_{-2,p}(x)+C_{-1}R_{-1,p}(x)+C_{1}R_{1,p}(x)+.......+C_{n-1}R_{n-1,p}(x)]=F(x)$$

i.e.

$$\begin{aligned}&[R''_{-2,p}(x)+k_1P(x)R'_{-2,p}(x)+k_2Q(x)R_{-2,p}(x)]C_{-2}+\\&[R''_{-1,p}(x)+k_1P(x)R'_{-1,p}(x)+k_2Q(x)R_{-1,p}(x)]C_{-1}+\\&[R''_{1,p}(x)+k_1P(x)R'_{1,p}(x)+k_2Q(x)R_{1,p}(x)]C_{1}.....\\&...\\&.......+[R''_{n-1,p}(x)+k_1P(x)R'_{n-1,p}(x)+k_2Q(x)R_{n-1,p}(x)]C_{n-1}=F(x)\end{aligned} \tag{7}$$

Now let the coefficients of C_{-2}, C_{-1}, C_1……, C_{n-1} are assumed as $R_{-2}(x)$, $R_{-1}(x)$, $R_1(x)$….$R_{n-1}(x)$, now we have the equation 7, as

$$[R_{-2}(x)]C_{-2}+[R_{-1}(x)]C_{-1}+[R_{1}(x)]C_{1}+.......+[R_{n-1}(x)]C_{n-1}=F(x) \tag{8}$$

In matrix form, we have

$$[R_{-2}(x)\quad R_{-1}(x)\quad R_{1}(x).\quad .\quad .\quad .\quad .R_{n-1}(x)]\begin{bmatrix}C_{-2}\\C_{-1}\\C_{1}\\.\\.\\.\\C_{n-1}\end{bmatrix}=F(x) \tag{9}$$

Equation (8) is evaluated at x_i's, i=1,2,3,….n-1 gives the system of (n-1)×(n+1) equations in which (n+1) arbitrary constants are involved.

The Matrix (9) can be written as

$$\begin{bmatrix}R_{-2}(1) & R_{-1}(1) & R_{1}(1)..... & R_{n-1}(1)\\R_{-2}(2) & R_{-1}(2) & R_{1}(2)..... & R_{n-1}(2)\\R_{-2}(3) & R_{-1}(3) & R_{1}(3)..... & R_{n-1}(3)\\. & . & . & .\\. & . & . & .\\. & . & . & .\\R_{-2}(n-1) & R_{-1}(n-1) & R_{1}(n-1)..... & R_{n-1}(n-1)\end{bmatrix}\begin{bmatrix}C_{-2}\\C_{-1}\\C_{1}\\.\\.\\.\\C_{n-1}\end{bmatrix}=\begin{bmatrix}F(1)\\F(2)\\F(3)\\.\\.\\.\\F(n-1)\end{bmatrix} \tag{10}$$

Applying boundary conditions to approximate the solution, we have

$$\sum_{i=-2}^{n-1} C_i R_{i,p}(a) = d_1$$

$$\sum_{i=-2}^{n-1} C_i R_{i,p}(b) = d_2 \tag{11}$$

A square matrix of size (n+1)x(n+1) is obtained from equations (10), (11)

i.e.,

$$\begin{bmatrix} R_{-2}(a) & R_{-1}(a) & R_1(a). & . & . & . & R_{n-1}(a) \\ R_{-2}(1) & R_{-1}(1) & R_1(1). & . & . & . & R_{n-1}(1) \\ R_{-2}(2) & R_{-1}(2) & R_1(2). & . & . & . & R_{n-1}(2) \\ R_{-2}(3) & R_{-1}(3) & R_1(3). & . & . & . & R_{n-1}(3) \\ . & . & .. & . & . & . & . \\ . & . & .. & . & . & . & . \\ . & . & .. & . & . & . & . \\ R_{-2}(n-1) & R_{-1}(n-1) & R_1(n-1). & . & . & . & R_{n-1}(n-1) \\ R_{-2}(b) & R_{-1}(b) & R_1(b). & . & . & . & R_{n-1}(b) \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ F(1) \\ F(2) \\ F(3) \\ . \\ . \\ . \\ F(n-1) \\ d_2 \end{bmatrix} \tag{12}$$

It is in the form of $[R][C]=[F]$

The matrix **[*R*]** is diagonally dominated square matrix of size (n+1) because of local support of basis functions. So, that the system of equations are easily solved for arbitrary constants C_i's.
We have

$$[C]=[F][R]^{-1} \tag{13}$$

The approximate solution becomes as known solution is obtained. Now the final approximation solution obtained by substituting these constants in equation(2). This approximate solution is used to evaluate the field variable at each node(Collocation point) in the considered domain.

IV. TEST PROBLEM

A numerical example is considered to study the efficiency and convergence of the Collocation Method.(Long cylindrical fin with insulated end)

Consider a steel rod of diameter D=0.02m, length L=0.05m, and thermal conductivity k=50(W/m.°C) is exposed to ambient air at T_∞=20°C with a heat transfer coefficient β=100(W/m^2.°C). The left end of the rod is maintained at temperature T_0=320°C and other end is insulated. Calculating the temperature distribution along the rod at different locations(nodes).

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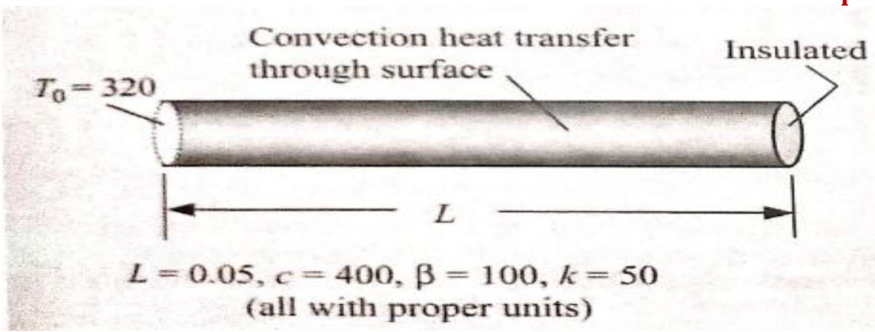

Figure 1: Cylindrical fin with insulated end.

Governing equation is $-\dfrac{d^2\theta}{dx^2} + m^2\theta = 0$ for $0 \le x \le L$ (14)

Where, $\theta=(T-T_\infty)$, temperature gradient and $m^2 = \dfrac{4\beta}{kD} = 400.$

With boundary conditions $\theta(0) = 300$ and $\dfrac{d\theta}{dx}(0.05) = 0$ and having a exact solution

$$\theta(x) = \frac{300 * \cosh(1-20x)}{\cosh(1)} \tag{15}$$

Comparing the given differential equation with equation (1)

$$\frac{d^2U}{dx^2} + k_1 P(x)\frac{dU}{dx} + k_2 Q(x)U = F(x)$$

Here F(x)= 0, a=0, b=0.05 and d_1=300, d_2=0, Q(x)=-400, U= θ

Taking the approximation function from the equation (2), it can be written as

$$\theta^h(x) = \sum_{i=-2}^{n-1} C_i R_{i,p}(x)$$

Taking number of intermittent segments (or sub domains) as 11 (i.e. n=11), order of NURBS curve as 3 (i.e. p=3). $X=\{a=X_1=0, X_2, \ldots X_{n-1}, X_n=b\}$ with non-uniform values between [a b], for homogeneous coordinates (weights) $h_{12}=0.75$, $h_i=1$ for $i\neq12$ and knot vector having 15 elements or knot values. Now the above equation can be modified as

$$\theta^h(x) = \sum_{i=-2}^{10} C_i R_{i,3}(x) \tag{16}$$

Substituting the approximation function in governing equation we have

$$\sum_{i=-2}^{10} C_i R''_{i,3}(x) - 400\sum_{i=-2}^{10} C_i R_{i,3}(x) = 0 \tag{17}$$

Expanding the above equation, we have

$$C_{-2}R''_{-2,3}(x) + C_{-1}R''_{-1,3}(x) + \ldots + C_9 R''_{9,3}(x) + C_{10}R''_{10,3}(x) - 400(C_{-2}R_{-2,3}(x)$$
$$+ C_{-1}R_{-1,3}(x) + \ldots + C_9 R_{9,3}(x) + C_{10}R_{10,3}(x)) = 0$$

i.e.,

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$$[R''_{-2,3}(x)-400R_{-2,3}(x)]C_{-2}+[R''_{-1,3}(x)-400R_{-1,3}(x)]C_{-1}.....$$
$$......+[R''_{10,3}(x)-400R_{10,3}(x)]C_{10}=0 \quad (18)$$

Here the functions $R_{i,3}(x)$ are defined by rational B-Spline basis functions and $R''_{i,3}(x)$ are the second derivative rational functions given from the equation.

In matrix form, we have from equation (9)

$$[R_{-2}(x) \quad R_{-1}(x) \quad R_1(x)...... \quad R_{10}(x)]\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{10} \end{bmatrix}=0 \quad (19)$$

Assuming where,

$$R''_{-2,3}(x)-400R_{-2,3}(x)=R_{-2}(x)$$
$$R''_{-1,3}(x)-400R_{-1,3}(x)=R_{-1}(x)$$
$$R''_{1,3}(x)-400R_{1,3}(x)=R_1(x)$$
$$.$$
$$.$$
$$R''_{10,3}(x)-400R_{10,3}(x)=R_{10}(x)$$

Equation(19) is evaluated at x_i ={0 0.0078 0.018 0.0213 0.0258 0.0278 0.0281 0.0347 0.0366 0.0418 0.05},for i=1,2,3,.....11 gives the system of (10)×(12) equations in which (12) arbitrary constants are involved.

i.e.,
$$\begin{bmatrix} R_{-2}(0) & R_{-1}(0) & R_1(0)..... & R_{10}(0) \\ R_{-2}(0.0078) & R_{-1}(0.0078) & R_{-1}(0.0078)..... & R_{10}(0.0078) \\ R_{-2}(0.018) & R_{-1}(0.018) & R_1(0.018)..... & R_{10}(0.018) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R_{-2}(0.0418) & R_{-1}(0.0418) & R_1(0.0418)..... & R_{10}(0.0418) \end{bmatrix}\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{10} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \quad (20)$$

Now let the approximation solution satisfy the boundary conditions. The given boundary conditions are from equation (4)

$$\sum_{i=-2}^{10} C_i R_{i,3}(0)=300$$

and

$$\sum_{i=-2}^{10} C_i R'_{i,3}(0.05) = 0 \tag{21}$$

Rewriting the matrix (20) using boundary condition equations (21), we get the system of (12) × (12) equations in which (12) arbitrary constants are involved. From matrix (20)

$$\begin{bmatrix} R_{-2,3}(0) & R_{-1,3}(0) & R_{1,3}(0)\ldots.. & R_{10,3}(0) \\ R_{-2}(0) & R_{-1}(0) & R_{1}(0)\ldots.. & R_{10}(0) \\ R_{-2}(0.006) & R_{-1}(0.006) & R_{1}(0.006)\ldots.. & R_{10}(0.006) \\ R_{-2}(0.012) & R_{-1}(0.012) & R_{1}(0.012)\ldots.. & R_{10}(0.012) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R_{-2}(0.048) & R_{-1}(0.048) & R_{1}(0.048)\ldots\ldots\ldots & R_{10}(0.048) \\ R'_{-2,3}(0.05) & R'_{-1,3}(0.05) & R'_{1,3}(0.05)\ldots\ldots & R'_{10,3}(0.05) \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ . \\ . \\ C_{10} \end{bmatrix} = \begin{bmatrix} 300 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \end{bmatrix} \tag{22}$$

It is in the form of $[R][C] = [F]$ (23)

Where $[R]$ is matrix of basis functions, $[C]$ is the matrix of arbitrary constants and $[F]$ is right side function matrix.

We have $[C] = [F][R]^{-1}$ (24)

By solving the set of equations (24) we get the constants C_i where i=1, 2, 3 …10. And by substituting these constant values in approximation solution equation then we get the final solution for the given problem.
Now the final approximation solution is evaluated at each node (Collocation point)i.e. x_i=0, 0.0078, 0.018, 0.0213 …0.0418 and the values field variable θ(x) at each node are calculated. The exact solution also evaluated at these points and result values of field variable θ(x) are compared with each other to find out the accuracy of the NURBS Collocation Method and shown in table 1.

Table 1: Comparison of field varible θ(x) with exact and NURBS Collocation Method with equal and unequal weights

Node(Knot Values)	Exact Sol.	NURBSCM Sol. with Equal weights, h_i=1 (B-Spline)	NURBSCM sol. with Unequal weights
0	300	300	300
0.0078	267.8705	266.9406	267.4089
0.018	235.6105	234.0371	235.1276
0.0213	227.3331	225.6986	226.9960
0.0258	217.6359	215.9371	217.5250
0.0278	213.8964	212.1777	213.8983
0.0281	213.3651	211.6435	213.3843
0.0347	203.5897	201.8194	204.0189

0.0366	201.4400	199.6677	202.0052
0.0418	197.0367	195.2576	197.9898
0.05	194.4163	192.6318	195.6852

From the above table 1, it can be stated that the values of the field variable obtained by NURBS Collocation Method using unequal weights are nearer to the Exact solution values. The absolute % of error at each node is calculated and the maximum absolute % of error is 0.4838.

Table 1 gives clear picture that the solution obtained from NURBS Collocation Method with unequal weights is much near to exact solution, when compared to solution obtained using equal weights. These values are plotted in figure 2. So it can be stated that the NURBS Collocation Method with unequal weights gives better approximation fit when compared to solution obtained by equal weights.

The error becomes negligible when we increase the no. of nodes i.e., collocation points. The absolute % of error at each node is calculated. The maximum absolute error is taken at different no. of nodes and given in table 2; these values are shown graphically in the figure 3.

Table 2: Maximum absolute % error when different no of nodes taken

S.no	No of Nodes	% of Error
1	10	0.4838
2	20	0.3954
3	30	0.1995
4	40	0.1351
5	50	0.0931

From the table 2 it can be stated that when the no. of Collocation nodes is increased the solution became very accurate as it is giving minimum absolute error. So that increasing the no. of collocation nodes gives convergence of this method. These values are plotted in figure 3.

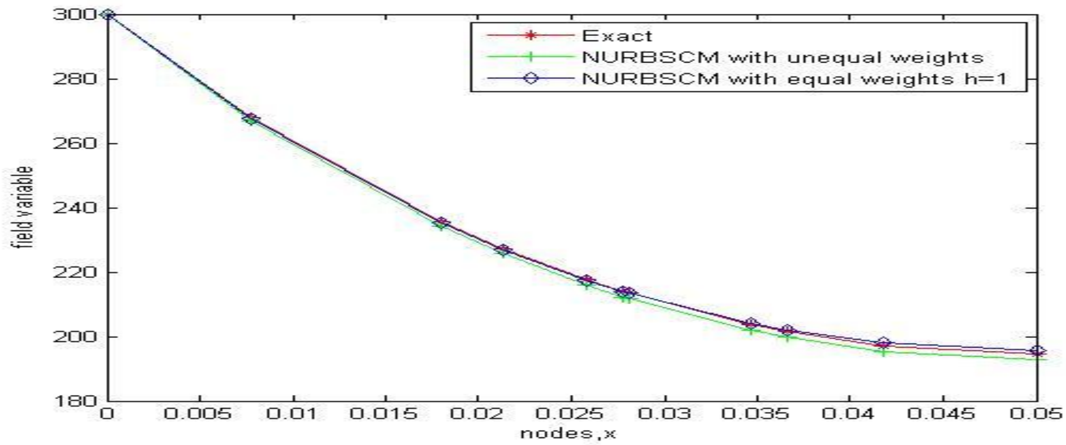

Figure 2: Comparison of field varible θ(x) with exact and NURBS Collocation Method with equal and unequal weights

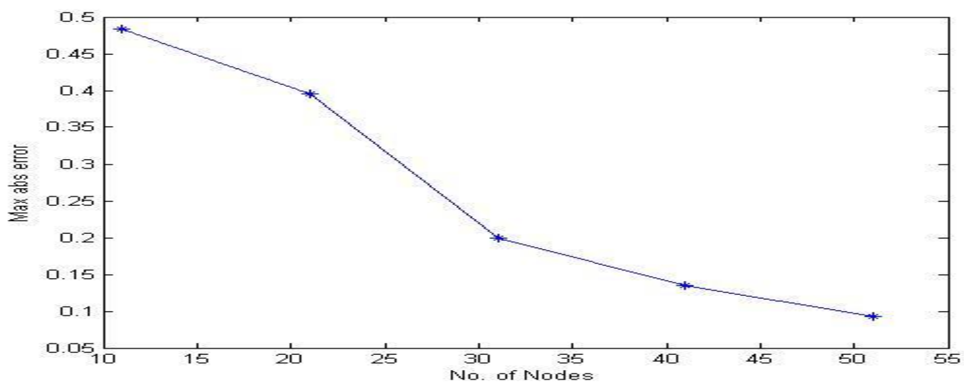

Figure 3:Maximum absolute % of error for different no of nodes.

Thus, it can be observed that the field variable obtained from the NURBS collocation method solution with unequal weights is in very good agreement with the exact solutions with less than 1% error when we increase no. of nodes.

V. CONCLUSION

In this paper, an attempt is made to use the NURBS basis functions as the shape functions in the finite element method. NURBS basis functions are defined recursively and incorporated in the collocation method. The accuracy and efficiency of the present method is illustrated by a heat transfer test problem. The NURBS Collocation Method solution is compared with exact solution and found to be in best fit approximation.

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