COMMON FIXED POINTS OF RELATIVELY NONEXPANSIVE MAPPINGS BY ITERATION

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Abstract
Let us consider two nonempty closed convex subsets A, B of a strictly convex space and \( f_i : A \cup B \to A \cup B, i = 1, 2, \ldots, k \) be relatively nonexpansive mappings. i.e. \( f_i(A) \subseteq A \) and \( f_i(B) \subseteq B \) and \( ||f_i(x) - f_i(y)|| \leq ||x - y|| \), for all \( x \in A \) and \( y \in B \). In this paper, we provide the strong convergence of some iteration of the mappings \( \{f_i\}_{k=1}^k \) to a common fixed point of \( \{f_i\}_{k=1}^k \) in strictly convex space setting, which generalizes a result of Kuhfittig [7].

Key words: Relatively nonexpansive mappings, fixed points.

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I. INTRODUCTION

We know that the behaviour of the iterated sequences play a role in fixed point theory. It is well known fact that if an iterated sequence of a continuous mapping \( T \) converges, then the limit of it must be a fixed point of \( T \). Also, Banach contraction principle states that every contraction mapping \( T : A \to A \), where \( A \) is a complete subspace of a metric space \( X \), has unique fixed point in \( A \) and every iterated sequence of \( T \) starting from any \( x \in A \) converges to the unique fixed point of \( T \). But the behaviour of the iterated sequences of nonexpansive mappings are completely different from the iterated sequences of contractive type mappings.

Consider a nonexpansive mapping \( T : A \to A \), where \( A \) is a nonempty closed convex subset of a normed linear space \( X \). In [1], Krasnoselskii proved that in uniformly convex Banach space \( X \), the sequence of successive approximation of the averaged mapping \( F : A \to A \) given by \( F(x) := (x + T x)/2 \), for all \( x \in A \), converges to a fixed point of the nonexpansive mappings \( T \). A complete proof of Krasnoselskii’s results in English can be found in [2]. Later, in [3], Edelstein extended Krasnoselskii’s result to strictly convex space setting.

In [4], the authors introduced a class of mappings called relatively nonexpansive defined as follows, which generalizes the notion of nonexpansive mappings.

DEFINITION 1. Let \( A, B \) be nonempty subsets of a normed linear space \( X \) and \( T : A \cup B \to A \cup B \) be a mapping. Then \( T \) is said to be a relatively nonexpansive mapping if and only if

1. \( T(A) \subseteq A \) and \( T(B) \subseteq B \),
2. \( ||T x - T y|| \leq ||x - y|| \), for all \( x \in A, y \in B \).

Define that \( \text{dist}(A, B) = \inf \{||a - b|| : a \in A, b \in B\} \) and for any given pair of subsets \( A, B \) of a normed linear space \( X \), define \( A_0 = \{x \in A; ||x - y|| = \text{dist}(A, B), \text{for some } y \in B\} \). In [5], the authors provided sufficient conditions.
which ensure the non emptiness of the set $A_0$. In [6], the authors proved that $A_0$ is contained in the boundary of the set $A$.

In [4], the authors introduced and used the geometric notion called proximal normal structure to prove the existence of the best proximity point. In [8], the authors generalized the results in [4]. In [7], the main result is as follows.

**THEOREM 1.1.** Let $C$ be a convex compact subset of a strictly convex Banach space $X$ and $\{T_i : i = 1, 2, \ldots, k\}$ a family of non-expansive self mappings of $C$ with a nonempty set of common fixed points.

Then for an arbitrary starting point $x \in C$, the sequence $\{U^n \}^\infty$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$.

In this article, we generalized the above theorem of [7].

**II. PRELIMINARIES**

In this section, we introduce basic definition and results which we used in our main result. We generalized the iteration of nonexpansive given in [7]

**REMARK 2.1.** Let $A, B$ be two nonempty convex subsets of a Banach space $X$. Let $f_i : A \cup B \to A \cup B, i = 1, 2, \ldots, k$, be a relatively nonexpansive mapping. Fix $F_0 = I$. For $0 < \alpha < 1$.

$$
F_1 = (1 - \alpha)I + \alpha f_1 F_0 \\
F_2 = (1 - \alpha)I + \alpha f_2 F_1 \\
\vdots \\
F_k = (1 - \alpha)I + \alpha f_k F_{k-1}.
$$

$x_{n+1} = (1 - \alpha) x_n + \alpha f_k F_{k-1} x_n$ \hspace{1cm} (1)

Put $k = 1$, $x_{n+1} = (1 - \alpha) x_n + \alpha f_1 F_0 x_n$ \hspace{1cm} (2)

Let us state an convergence result, which plays a vital role in our main result.

**THEOREM 2.1.** [8]Let $A, B$ be nonempty closed convex subsets of a strictly convex Banach space $X$ such that $A_0$ is nonempty. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose $T (A)$ is contained in a compact subset $A_1$ of $A$. Then the Krasnoselskii’s iteration $\{F^n (x)\}$, where $F : A \cup B \to A \cup B$ given by $F (x) = \frac{1}{2} (T x + x)$, converges to a fixed point of $T$.

**III. MAIN RESULT**

Our main result is as follows.

**THEOREM 3.1.** Let $A, B$ be two nonempty convex, compact subsets of a strictly convex Banach space $X$ with $A_0$ is nonempty. Let $f_i : A \cup B \to A \cup B, i = 1, 2, 3 \ldots, k$ be mappings with a non empty set of fixed points $\|f_i (x)\|$
Proof. We can easily prove that the mappings $F_j$ and $f_jF_{j-1}$, $j = 1, 2, \ldots, k$ are relatively nonexpansive and map $A \cup B$ into itself.

Now we are going to prove \{${F_k^n(x)}$\} converges to a fixed point of $f_i$, $\forall x \in A \cup B$.

Let $x \in A \cup B$ with $f_j(x) = x$, $j = 1, 2, \ldots, k$.

Then

$$F_1(x) = (1 - \alpha)x + \alpha f_1F_0(x) = (1 - \alpha)x + \alpha f_1(x) = (1 - \alpha)x + \alpha x = x,$$

$$F_2(x) = (1 - \alpha)x + \alpha f_2F_1(x) = x$$

Proceeding like this, we get $F_j(x) = x$, $j = 1, 2, \ldots, k$.

Now, let $F_j(x) = x$, $j = 1, 2, \ldots, k$.

$$x = F_j(x) = (1 - \alpha)x + \alpha f_jF_{j-1}(x) = (1 - \alpha)x + \alpha f_j(x)$$

$$\Rightarrow \alpha x = \alpha f_j(x)$$

Hence $f_j(x) = x$, $j = 1, 2, \ldots, k$.

Since (1) has the same form as (2), \{${F_k^n(x)}$\} converges to a fixed point $y$ of $f_kF_{k-1}$. We wish to show next that $y$ is a common fixed point of $f_k$ and $F_{k-1}$ ($k \geq 2$). To this we first show that $f_{k-1}F_{k-2}y = y$. Suppose not, the closed line segment $[y, f_{k-1}F_{k-2}y]$ has positive length.

Let $z = f_{k-1}y = (1 - \alpha)y + \alpha f_{k-1}F_{k-2}(y)$

By hypothesis, there exists a point $w \in A \cup B$ such that $f_1w = f_2w = \ldots = f_kw = w$. Since $f_i$ and $F_i$ have the same common fixed points, it follows that $f_{k-1}F_{k-2}w = w$.

By relatively nonexpansive, $\|f_{k-1}F_{k-2}y - w\| \leq \|y - w\|$ and $\|f_kw - w\| \leq \|z - w\|$ (3)

So $w$ is atleast as close to $f_kw$ as to $z$.

But $f_kw = f_kF_{k-1}y = y$. Therefore $w$ is atleast as close to $y$ as to $z = (1 - \alpha)y + \alpha f_{k-1}F_{k-2}y$.

Since $X$ is strictly convex, $\|y - w\| < \|f_{k-1}F_{k-2}y - w\|$, which is a contradiction to (3). Therefore $f_{k-1}F_{k-2}y = y$.

Now, $F_{k-1} = (1 - \alpha)y + ay = y$ and $y = f_{k-1}F_{k-2}y = f_ky$

$\Rightarrow y$ is a common fixed point of $f_k$ and $F_{k-1}$. Repeating the argument, we conclude that $y$ is a common fixed point of $f_i$ and $F_j$, $j = 1, 2, \ldots, k$.

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